Singularity of Cannon–Thurston maps

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Abstract

In a closed fibered hyperbolic 3-manifold M, the inclusion of a fiber S, with S and M lifted to the universal covers $\widetilde{S}$ and $\widetilde{M}$, gives an exponentially distorted embedding of the hyperbolic plane into hyperbolic 3-space. Nevertheless, Cannon and Thurston showed that there is a map from the circle at infinity of the hyperbolic plane to the 2-sphere at infinity of hyperbolic 3-space. The Cannon–Thurston map is surjective, finite-to-one, and gives a space-filling curve.

Here we prove that many natural measures on the circle when pushed forward by the Cannon–Thurston map become singular with respect to many natural measures on the 2-sphere. The circle measures we consider are the Lebesgue measure and stationary measures that arise from fully supported random walks on the surface group. Whereas the measures on the sphere we consider are the Lebesgue measure and stationary measures that arise from geometric random walks on the 3-manifold group.

The singularity of measures is ultimately derived from the following geometric result. We prove that a hyperbolic geodesic sampled with respect to a pushforward measure asymptotically spends a definite proportion of its time close to a fiber. On the other hand, we show that a hyperbolic geodesic sampled with respect to a natural measure on the sphere spends an asymptotically negligible proportion of its time close to a fiber. For a more restricted class of circle measures, namely the Lebesgue measure and stationary measures from geometric random walks on the surface group, we also prove an effective result for the proportion of time spent close to a fiber. To this end, we give precise descriptions of quasi-geodesics in the Cannon–Thurston metric, which may be of independent interest.

Contents

1 Introduction

Let M be a closed 3-manifold that fibers over the circle. Suppose that the fiber S is a closed orientable surface. Fixing a fiber, M is homeomorphic to a mapping torus $S \times [0,1]/\sim$, where $\sim$ is the identification of $S \times \{0\}$ with $S \times \{1\}$ by a diffeomorphism $f : S \to S$. The topology of M depends only on the isotopy class of f, that is, only on f as a mapping class.

Thurston's hyperbolization theorem states that such a manifold M admits a complete hyperbolic metric if and only if S has negative Euler characteristic and the monodromy $f : S \to S$ is pseudo-Anosov. In this case, the universal covers of S and M can be identified with $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. The natural inclusion of a fiber as $S \times \{0\}$ induces a map $\iota : \widetilde{S} = \mathbb{H}^2 \hookrightarrow \mathbb{H}^3 = \widetilde{M}$ between the universal covers. The inclusion ι is exponentially distorted for the hyperbolic metrics on the source and the target. Cannon–Thurston proved that despite the distortion, the inclusion extends to a $\pi_1(S)$-equivariant, continuous map at infinity, known as the Cannon-Thurston map, which we shall also call ι, i.e. $\iota : \partial\mathbb{H}^2 = S^1_\infty \to S^2_\infty = \partial\mathbb{H}^3$ [CT07]. We maintain this notation of M, S, f, and ι throughout this paper. In particular, we assume throughout that $\chi(S) < 0$ and f is pseudo-Anosov.

We may compare measures on S^1_∞ with measures on S^2_∞, by taking the pushforward of the measures on S^1_∞ under the Cannon-Thurston map. We now describe the collections of measures that we will consider. The first collection, which we refer to as *surface measures*, are measures on the boundary of the universal cover of the surface S, namely S^1_∞. We write S_h for S endowed with a particular choice of hyperbolic metric. That is, S_h is a hyperbolic structure.

Suppose that G is a group. A probability measure μ on G generates a random walk on G. The steps of the random walk are independent identically μ-distributed random variables $(g_n) \in (G, \mu)^{\mathbb{N}}$, and the location of the random walk at time n is given by $w_n = g_1 \dots g_n$. If G acts on a metric space (X, d) with a basepoint x_0, we say that μ has *finite exponential moment* if there is a constant $c > 1$ such that $\sum_{g \in G} \mu(g) c^{d(x_0, gx_0)}$ is finite. We say a probability measure μ on a group G is *geometric* if μ has finite exponential moment with respect to a word metric on G and if the support of μ generates G as a semigroup. We denote the set of geometric probability measures on G by $\mathcal{P}(G)$. In this paper, the group G will be either $\pi_1(S)$ or $\pi_1(M)$. In these cases, for any basepoint x_0, almost every sample path $w_n x_0$ of the random walk converges to the boundary, and the resulting boundary measure is known as the *hitting measure* determined by μ. For any two basepoints x_0 and x_1, the distance between $w_n x_0$ and $w_n x_1$ is constant, so the hitting measure does not depend on the choice of basepoint.

Definition 1. Let S_h be a closed hyperbolic surface of genus at least 2. Let S^1_∞ be the Gromov boundary of the universal cover, with basepoint x_0. We will refer to the following measures on S^1_∞ as *geometric surface measures.*

- The Lebesgue measure on S^1_∞ determined by the visual measure at x_0.
- The hitting measure on S^1_∞ for a random walk determined by a geometric probability distribution $\mu \in \mathcal{P}(\pi_1(S))$.

The Lebesgue measures that arise as the visual measures from different marked hyperbolic metrics are mutually singular, see for example Agard [Aga85]. However, for a fixed hyperbolic metric, different choices of basepoints give absolutely continuous measures. Similarly, different choices of the probability measure μ on $\pi_1(S)$ are expected to usually give mutually singular hitting measures, whereas for a fixed μ, the hitting measures arising from different basepoint choices are absolutely continuous with respect to each other.

It is a long standing conjecture of Guivarc'h–Kaimanovich–Ledrappier (see [DKN09, Conjecture 1.21]) that a hitting measure ν that arises from any finitely supported random walk on $\pi_1(S)$ is singular with respect to the Lebesgue measure on S^1_∞. Thus, Definition 1 covers a wide class of measures.

It will also be convenient to consider random walks on surface groups generated by more general probability measures. We say a probability measure μ on $\pi_1(S)$ is *nonelementary* if the semigroup generated by the support of μ contains a pair of non-trivial elements with disjoint fixed points on the boundary S^1_∞. We say a probability measure μ on $\pi_1(S)$ is *full* if every open set in S^1_∞ has positive hitting measure.

Definition 2. Let S be a closed hyperbolic surface of genus at least 2. Let S^1_∞ be the Gromov boundary of the universal cover. We refer to the larger collection of measures on S^1_∞ which contains in addition to geometric surface measures,

- the hitting measure on S^1_∞ for a random walk determined by a nonelementary, full probability measure μ on $\pi_1(S)$.

as the *full surface measures*.

The final collection of measures which we shall refer to as 3-*manifold measures*, are measures on the boundary of the universal cover of the 3-manifold M, namely S^2_∞.

Definition 3. Let M be a closed hyperbolic 3-manifold. Let S^2_∞ be the Gromov boundary of the universal cover, with basepoint x_0. We shall call the following measures 3-*manifold measures*.

- The Lebesgue measure on S^2_∞ determined by the visual measure at x_0.
- The hitting measure on S^2_∞ determined by a geometric random walk generated by $\mu \in \mathcal{P}(\pi_1(M))$.

As with the Lebesgue measures on S^1_∞, Lebesgue measures on S^2_∞ are expected to be mutually singular with respect to the hitting measures arising from finitely supported random walks on $\pi_1(M)$. The Lebesgue measures arising from different basepoints are absolutely continuous with respect to each other. The hitting measures arising from different probability measures μ are typically expected to be mutually singular.

Let M be a closed hyperbolic 3-manifold which fibers over the circle. As mentioned before, this determines a collection of 3-manifold measures on S^2_∞ given by Definition 3. Let S be a fiber and let S_h be a hyperbolic structure on S. These choices of M and S_h determine a collection of surface measures on S^1_∞, given by Definition 1. The pushforwards of these surface measures by the Cannon-Thurston map ι give measures on S^2_∞, so that we may now compare the surface measures to the 3-manifold measures:

Theorem 4. *Suppose that M is a closed hyperbolic 3-manifold that fibers over the circle. Suppose that S_h is a hyperbolic structure on a fiber S. Then pushforwards to S^2_∞ under the Cannon-Thurston map of any of the surface measures from either Definition 1 or Definition 2 are mutually singular with respect to any of the 3-manifold measures from Definition 3.*

Suppose now that M is a compact hyperbolic 3-manifold and S is an incompressible surface in M. As an immediate corollary of Theorem 4, we obtain:

Corollary 5. *Suppose that μ is a geometric probability measure on the fundamental group of any incompressible surface S of a closed hyperbolic 3-manifold M. Then the hitting measure on S^2_∞ arising from μ is mutually singular with respect to both the Lebesgue measure on S^2_∞ and any hitting measure arising from a geometric random walk on $\pi_1(M)$.*

Proof. An incompressible surface S in M is either quasi-Fuchsian or a virtual fiber, by the Tameness Theorem [CG06] and the Covering Theorem [Can96]. If it is quasi-Fuchsian then the hitting measure is supported on the limit set of the surface subgroup. This limit set has measure zero for both the Lebesgue measure on S^2_∞ and for any hitting measure arising from a geometric random walk on $\pi_1(M)$.

On the other hand, if S is a virtual fiber then M admits a fibered finite cover in which the fiber F is a finite cover of S. As $\pi_1(F)$ is a finite index subgroup of $\pi_1(S)$, a geometric random walk on $\pi_1(S)$ is recurrent on $\pi_1(F)$, and the restriction of the random walk on $\pi_1(S)$ to $\pi_1(F)$ is the random walk on $\pi_1(F)$ generated by the first hitting measure μ' of the random walk of $\pi_1(S)$ on $\pi_1(F)$. The probability measure μ' is not finitely generated, but is nonelementary, and has the same hitting measure as μ, so is full. Therefore, Theorem 4 implies that the hitting measure is mutually singular with respect to either of the 3-manifold measures on S^2_∞. □

Singularity of the pushforward of the Lebesgue measure on S^1_∞ with respect to the Lebesgue measure on S^2_∞ was previously shown by Tukia (see [Tuk89, page 430]) using conformal techniques, such as cross-ratios. Kim and Zimmer give further such rigidity results [KZ25]. As we describe in the next section, Section 1.1, we give an argument using properties of typical geodesics chosen according to the measures on the boundary, which enables us to extend the results to hitting measures arising from random walks. In light of the Guivarc'h–Kaimanovich–Ledrappier conjecture, we do not expect hitting measures to be conformal.

1.1 Statistics for typical geodesics

An oriented geodesic in $\mathbb{H}^n$ is determined by its (ordered) endpoints in $(S^{n-1}_\infty \times S^{n-1}_\infty) \setminus \Delta$, where Δ is the diagonal. A probability measure ν on S^{n-1}_∞ gives rise to a probability measure $\nu \times \nu$ on the space of oriented geodesics in $\mathbb{H}^n$, as long as the diagonal has measure zero.

For random walks, if μ is not symmetric, we will use the measure $\nu \times \check{\nu}$ instead of the product measure, where $\check{\nu}$ is the hitting measure corresponding to the limits $\lim_{n \to -\infty} w_n x_0$ of the sample paths. To prove Theorem 4, we show the desired measures are mutually singular by showing that they give rise to typical geodesics possessing different behavior.

The fibration of M by closed surfaces lifts to a fibration of the universal cover by the universal covers of the fibers. We will call the image of $\widetilde{S}$ under the Cannon-Thurston map the *base fiber* $S_0 := \iota(\widetilde{S})$.

- With respect to the pushforwards of the surface measures to S^2_∞, almost all geodesics in $\mathbb{H}^3$ spend a positive proportion of time close to the base fiber S_0.
- With respect to the 3-manifold measures on S^2_∞, for almost all geodesics in $\mathbb{H}^3$, the proportion of time the geodesic spends close to the base fiber S_0 tends to zero.

We now give precise versions of these statements. Let $\gamma(t)$ be a geodesic with unit speed parametrization. We will write $\gamma([0,T])$ for the segment of γ between $\gamma(0)$ and $\gamma(T)$. For any subset A of γ, we will write $|A|$ for the standard Lebesgue measure of A. The definition of the Cannon-Thurston metric depends on a choice of a hyperbolic structure S_h on S, and a pair of invariant measured laminations Λ_+ and Λ_- for f. We will write (S_h, Λ) to denote a choice of a hyperbolic metric and pair of invariant measured laminations on S_h. For surface measures we show:

Theorem 6. *Suppose that $f \colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic structure on S together with a pair of invariant measured laminations. Let $\widetilde{S}_h \times \mathbb{R}$ be the universal cover of the corresponding mapping torus, and let ι be the Cannon-Thurston map. Let ν be one of the surface measures from either Definition 1 or Definition 2.*

Then there are constants $R \geqslant 0$ and $\epsilon > 0$ (that depend on f and ν) such that for $\iota_*\nu$-almost all geodesics γ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\gamma(t)$,

$$\lim_{T \to \infty} \frac{1}{T} |\gamma([0,T]) \cap N_R(S_0)| \geqslant \epsilon.$$

In fact, the constant ϵ tends to one as R tends to infinity. Restricting the surface hitting measures to those arising from geometric random walks, we prove the following effective bound on the rate of convergence.

Theorem 7. *Suppose that $f \colon S \to S$ is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic structure on S together with a pair of invariant measured laminations. Let $\widetilde{S}_h \times \mathbb{R}$ be the universal cover of the corresponding mapping torus, and let ι be the Cannon-Thurston map. Let ν be one of the surface measures from Definition 1.*

Then there are constants $K > 0$ and $\alpha > 0$ such that for $\iota_*\nu$-almost all geodesics γ in $\widetilde{S}_h \times \mathbb{R}$ and for any unit speed parametrization $\gamma(t)$, for any $R \geqslant 0$,

$$\lim_{T \to \infty} \frac{1}{T} |\gamma([0,T]) \cap N_R(S_0)| \geqslant 1 - K e^{-\alpha k^R}.$$

For 3-manifold measures we show:

Theorem 8. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic structure on S together with a pair of invariant measured laminations. Let $\widetilde{S}_h \times \mathbb{R}$ be the universal cover of the corresponding mapping torus and let ι be the Cannon-Thurston map. Let ν be one of the 3-manifold measures from Definition 3.*

Then for ν-almost all geodesics γ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\gamma(t)$ and any $R > 0$,

$$\lim_{T\to\infty} \frac{1}{T} |\gamma([0,T]) \cap N_R(S_0)| = 0.$$

The mutual singularity of the surface measures and the 3-manifold measures, given in Theorem 4, is an immediate consequence of Theorem 6 and Theorem 8. Sets exhibiting the mutual singularity may be explicitly described as sets of geodesics which spend a positive proportion of time close to the base fiber S_0, and sets of geodesics whose proportion of time close to S_0 tends to zero.

Compared to Theorem 6, the effective version, namely Theorem 7, relies on an explicit construction of certain quasi-geodesics which we now briefly describe, see Section 6 for more details.

For a compact hyperbolic 3-manifold M which fibers over the circle, the universal cover $\widetilde{M}$ has two natural structures. First, the hyperbolic metric on M lifts to a hyperbolic metric on $\widetilde{M}$, which is isometric to $\mathbb{H}^3$. Any $\pi_1(M)$-invariant metric on $\widetilde{M}$ will be quasi-isometric to this hyperbolic metric. Second, as M fibers over the circle, the fibration lifts to a product structure $\widetilde{S} \times \mathbb{R}$ on $\widetilde{M}$. This is only a topological product structure, as no product metric on $\widetilde{S} \times \mathbb{R}$ can be quasi-isometric to the hyperbolic metric.

Given a hyperbolic structure S_h, the pseudo-Anosov mapping class f determines a pair of invariant measured laminations. A choice of a hyperbolic structure S_h lifts to a hyperbolic structure $\widetilde{S}_h$ on the universal cover $\widetilde{S}$, and the invariant measured laminations can be realized as geodesic measured laminations in this metric. Cannon and Thurston [CT07] used the invariant geodesic measured laminations to construct a $\pi_1(M)$-invariant pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$, which is therefore quasi-isometric to the hyperbolic metric on $\widetilde{M}$, see Section 2.3 for further details. We call this pseudo-metric the *Cannon-Thurston metric* on $\widetilde{S}_h \times \mathbb{R}$.

Lifting an invariant lamination to $\widetilde{S}_h$, any leaf ℓ of the lift is geodesic with respect to the hyperbolic metric on $\widetilde{S}_h$. However, its image $\iota(\ell)$ in $\widetilde{S}_h \times \mathbb{R}$ is embedded metrically as a horocycle. In particular, $\iota(\ell)$ is not quasigeodesic, and a segment of $\iota(\gamma)$ of length L has endpoints distance roughly $\log L$ apart in the Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$. For any surface measure, almost all geodesics γ in $\widetilde{S}_h$ have arbitrarily long subsegments that fellow travel leaves of the invariant laminations. The image $\iota(\gamma)$ in $\widetilde{S}_h \times \mathbb{R}$ is thus not quasigeodesic, even up to reparametrization. Our basic idea is to straighten "horocyclic" $\iota(\gamma)$ segments, that is, replace γ-subsegments that fellow-travel leaves of the invariant laminations by shortcuts. To do so, we define a *height function* $h_\gamma(t)$ along the geodesic, which is roughly $\log\log$ of the distance (in the unit tangent bundle) from the geodesic to the invariant laminations. We then show that the paths $(\gamma(t), h_\gamma(t))$ in $\widetilde{S}_h \times \mathbb{R}$ are uniformly quasigeodesic, i.e. their quasigeodesic constants do not depend on the choice of the geodesic γ in $\widetilde{S}_h$.

The pseudo-Anosov map f also determines a singular flat metric S_q on the fiber, such that the horizontal and vertical foliations are invariant measured foliations for f. There is an analogous Cannon-Thurston metric to that on $\widetilde{S}_q \times \mathbb{R}$ defined on the universal cover of the 3-manifold thought of as $\widetilde{S}_q \times \mathbb{R}$, so called the *singular solv metric*. Previously, McMullen [McM01] constructed explicit quasigeodesics in $\widetilde{S}_q \times \mathbb{R}$, using saddle connections in the flat metric. Even though the Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ and the singular solv metric on $\widetilde{S}_q \times \mathbb{R}$ are quasi-isometric, we do not know how to directly deduce our results from his construction. We give a detailed discussion of McMullen's work and its relation to ours in Section 2.10.

In Section 2, we review some previous results we use, and define some notation. In Section 3, we prove Theorems 6 and 8, which immediately imply the singularity of measures result, Theorem 4. In Section 4, we prove the effective bounds in Theorem 7, by assuming the main result of Section 6. In Section 5, we review some more results and definitions we need, and then in Section 6, we define the height function, and show that the paths specified by the height function are quasigeodesics.

1.2 Remarks on Related Work

It is interesting to compare Cannon–Thurston maps with the more classical space filling curves, such as the Peano curve. In contrast to our results here, the Peano curve is absolutely continuous.

The Peano curve is also Hölder with exponent 1/2. In contrast, Miyachi proved that Cannon–Thurston maps are not Hölder, see [Miy06, Theorem 1.1]. Since non-Hölder maps can be absolutely continuous, the absence of regularity does not imply singularity for the pushforward of the Lebesgue measure on the circle.

There are many Cannon–Thurston type phenomena generalizing the setup from fibered hyperbolic 3-manifolds. For some examples, see [GH24]. In the case of Gromov hyperbolic extensions of surface groups, we expect pushforwards of stationary measures on the circle to be singular with respect to geometric/fully supported stationary measures on the Gromov boundary of the extension. On the one hand, we expect geodesics in the extension sampled by the pushforward measure to spend a definite proportion of their time in a neighborhood of the surface subgroup. On the other hand, we expect geodesics sampled by stationary measures resulting from geometric/fully supported random walks to spend asymptotically negligible time in a neighborhood of the surface subgroup. While the reasons for our expectations are similar, our methods here are specific for hyperbolic 3-manifolds and these questions are left for now to future work.

For Kleinian surface groups, one obtains Cannon-Thurston maps more generally from doubly degenerate surface group representations in $\mathrm{PSL}(2,\mathbb{C})$ [Mj14]. Without the $\mathbb{Z}$-periodicity, it makes good sense only to compare the pushforward of the Lebesgue measure on S^1_∞ with the Lebesgue measure on S^2_∞, and here the singularity of the pushforward is again covered by Tukia's work. We indicate which of our techniques underlying Theorem 6 and Theorem 8 hold when the $S \times \mathbb{R}$ has bounded geometry; we leave the full discussion of our perspective in the bounded geometry case to future work. We also leave more general Cannon–Thurston situations to future work.

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Part I
Singularity of measures

2 Background

In this section, we review some background and fix notation.

Let S be a closed orientable surface of genus at least two and suppose that $f: S \to S$ is an orientation preserving diffeomorphism. Then f determines a mapping torus $M_f = S \times [0,1]/\sim$, which is the quotient of $S \times [0,1]$ by the relation $(p,0) \sim (f(p),1)$. The map f is often referred to as the *monodromy map* for the mapping torus. We say a closed orientable 3-manifold M *fibers over the circle* if M is homeomorphic to a mapping torus M_f of some orientation preserving surface diffeomorphism $f: S \to S$. Thurston [Thu22] showed that the 3-manifold M_f is hyperbolic if and only if f is pseudo-Anosov. We will always assume that f is pseudo-Anosov and so M_f is hyperbolic. For simplicity of notation, we will just write M for M_f.

The universal cover $\widetilde{M}$ may be thought of as $\mathbb{R}^2 \times \mathbb{R}$, where

- the universal cover $\widetilde{S}$ of the fiber S is identified with $\mathbb{R}^2$ and
- the monodromy map f acts by unit translation in the $\mathbb{R}$-factor direction.

Since f is pseudo-Anosov, the action of f on the Teichmüller space (of marked complete hyperbolic metrics on S) has an invariant axis on which f acts by translation. By identifying the $\mathbb{R}$ factor in $\widetilde{S} \times \mathbb{R}$ with the Teichmüller axis, we obtain a marked hyperbolic metric on S corresponding to $\widetilde{S} \times \{0\}$. We shall denote this metric by S_h. The monodromy map f acts by a change of marking of S_h. The universal cover $\widetilde{S}_h$ is isometric to the hyperbolic plane $\mathbb{H}^2$. We will write $d_{\mathbb{H}^2}$ for the hyperbolic metric on $\widetilde{S}_h$. The inclusion map ι sends $\widetilde{S}_h$ to $\widetilde{S}_h \times \{0\}$ in the universal cover $\widetilde{S}_h \times \mathbb{R}$ of M. We denote $\iota(\widetilde{S}_h)$ by S_0.

As in Section 1.1, the action of f on the fiber S_h has a pair of invariant measured geodesics laminations. The transverse measures are uniquely ergodic and the laminations are transverse to each other. The monodromy map f acts by stretching the leaves of one lamination (called the *unstable lamination*) and by contracting the leaves of the transverse lamination (called the *stable lamination*). The laminations can be lifted to $\pi_1(S)$-equivariant laminations of $\widetilde{S}_h$. These laminations reappear with further details in Section 2.1.

Using the lifted laminations, Cannon and Thurston [CT07] constructed a $\pi_1(M)$-invariant pseudometric on the universal cover $\widetilde{S}_h \times \mathbb{R}$ of M. See Section 2.3 for further details about the *Cannon–Thurston metric*.

The fiber lying on the invariant Teichmüller axis also carries a singular flat metric given by the associated quadratic differential on the underlying marked conformal surface. The monodromy map f acts affinely on the singular flat metric stretching the horizontal foliation and contracting (by the reciprocal of the stretch factor) the vertical foliation. We call the singular flat metric the *invariant flat metric* for f.

In an analogous way to Cannon–Thurston, we can use the invariant flat metric to construct a $\pi_1(M)$-equivariant metric on the universal cover $\widetilde{S}_q \times \mathbb{R}$ of M. This metric is known as the *singular solv metric*.

By the Švarc–Milnor lemma, both metrics, and also the hyperbolic metric, on $\mathbb{H}^3$ are quasi-isometric to $\pi_1(M)$ and so quasi-isometric to each other.

We do not use the singular solv metric directly, but we do discuss the relation between the Cannon-Thurston metric and the singular solv metric in Section 2.9. We shall always write $\widetilde{S}_h \times \mathbb{R}$ for the universal cover of M to emphasize that we are using the Cannon-Thurston metric and not the others.

2.1 Measured laminations

The properties of measured laminations that we present below are standard, see e.g. [CB88]. We present them in detail to keep our discussion self-contained.

A (possibly bi-infinite) geodesic on S_h is *simple* if it has no self-intersections. A *geodesic lamination* on S_h is a closed union of simple pairwise disjoint geodesics. A *transverse measure* on a geodesic lamination Λ is a positive measure dm defined on local transverse arcs to the leaves of Λ that

- is invariant under any isotopy preserving the transverse intersections with the leaves of Λ, and
- is positive and finite on any nontrivial compact transversals.

Such a measure lifts to a $\pi_1(S)$-invariant transverse measure on the pre-image of Λ in $\mathbb{H}^2$. By abuse of notation, we will denote the pre-image also by Λ and the lifted measure also by dm. A geodesic lamination equipped with a transverse measure is called a *measured lamination*. We will only consider measured laminations, and so we will often just write lamination to mean measured lamination.

We say a measured lamination is *filling* if there are no essential simple closed curves disjoint from the lamination. The complement of a filling lamination is a union of ideal polygons with finitely many sides. We say a leaf of the lamination is a *boundary leaf* if it is the boundary of an ideal polygon. There are only finitely many ideal complementary regions in the compact surface S_h, and so there are only finitely many boundary leaves in S_h. This implies that there are countably many boundary leaves in the universal cover $\widetilde{S}_h$.

We say a measured lamination is *minimal* if every leaf is dense in the lamination. A minimal filling lamination has the following properties:

- there are uncountably many leaves,
- no leaf is isolated, and
- the transverse measure is non-atomic.

We say a pair of measured laminations Λ_+ and Λ_- *bind* S_h if each geodesic ray on S_h crosses a leaf of $\Lambda_+ \cup \Lambda_-$.

Definition 9. We shall write (S_h, Λ) for a triple consisting of a hyperbolic metric on a compact surface S, together with a pair of minimal filling measured laminations Λ_+ and Λ_- which bind the surface. We refer to such a triple (S_h, Λ) as a *hyperbolic surface and a full pair of laminations*.

A pseudo-Anosov map f determines a pair of invariant measured laminations called stable and unstable laminations, which we shall denote (Λ_+, dx) and (Λ_-, dy) respectively, where dx and dy are the transverse measures. In particular,

- $f(\Lambda_+) = \Lambda_+$ and $f_* dx = k dx$, and
- $f(\Lambda_-) = \Lambda_-$ and $f_* dy = k^{-1} dy$,

where $k = k_f > 1$ is known as the *stretch factor* of the pseudo-Anosov map. We shall often write Λ_+ or Λ_- to refer to the measured laminations if we do not need to refer to the respective measures. In the coordinate system described in Section 2.3, leaves of Λ_- correspond to lines parallel to the x-axis, and leaves of Λ_+ correspond to lines parallel to the y-axis.

Cannon and Thurston [CT07, Theorem 10.1] showed that the invariant measured laminations Λ_+ and Λ_- are each minimal and filling, and together they *bind* the surface S_h. In fact, pseudo-Anosov invariant laminations are uniquely ergodic, that is, the transverse measures are unique up to scale. In particular, the invariant measured laminations Λ_+ and Λ_- form a *full pair* of laminations for S_h.

We now record some useful properties of full pairs of laminations.

Proposition 10. *Let (S, Λ) be a hyperbolic surface and a full pair of laminations. Then Λ_+ and Λ_- have no leaf in common.*

Proof. Suppose that Λ_+ and Λ_- share a leaf ℓ. Since each lamination is minimal, the common leaf ℓ is dense in both laminations. This implies that $\Lambda_+ = \Lambda_-$. But then any ideal complementary region contains a geodesic ray disjoint from both laminations, contradicting the fact that Λ_+ and Λ_- bind the surface. □

In Proposition 11, we combine the compactness of S with the absence of common leaves to deduce the following properties: first, there is a lower bound on the angle at any point of intersection of a leaf of Λ_+ with a leaf of Λ_-, second, if a leaf of Λ_+ in $\widetilde{S}_h$ is disjoint from a lift of leaf of Λ_- in $\widetilde{S}_h$, then there is a lower bound on the distance between them. The second property implies that the ideal complementary regions of one lamination do not share (ideal) vertices with the ideal complementary regions of the other lamination. Finally, there is an upper bound on the length of a segment of a leaf which does not intersect the other lamination.

Suppose ℓ and ℓ' are leaves of Λ_+ and Λ_- (in $\widetilde{S}_h$) that intersect, creating two pairs of complementary angles. We define the *angle of intersection* to be the smallest of the two angles at the point of intersection.

Proposition 11. *Suppose that (S_h, Λ) is a hyperbolic surface together with a full pair of measured laminations. Then there exist constants $\alpha_\Lambda, \epsilon_\Lambda, L_\Lambda > 0$ such that each of the following properties holds for any pair of leaves (ℓ_+, ℓ_-) with ℓ_+ in Λ_+ and ℓ_- in Λ_-:*

(11.1) If ℓ_+ and ℓ_- intersect, then their angle of intersection is at least α_Λ.

(11.2) If ℓ_+ and ℓ_- are disjoint, the distance between any two lifts of ℓ_+ and ℓ_- in $\widetilde{S}_h$ is at least ϵ_Λ.

(11.3) Any segment of a leaf of one of the laminations of length at least L_Λ intersects a leaf of the other lamination.

(11.4) No ideal complementary region in $\widetilde{S}_h \setminus \Lambda_+$ has an ideal vertex in common with an ideal complementary region of $\widetilde{S}_h \setminus \Lambda_-$.

Proposition (11.4) follows directly from Proposition (11.2); we prove the remaining statements.

Proof of Proposition (11.1). Suppose that there is a sequence (ℓ_n^-, ℓ_n^+) of pairs of intersecting leaves whose angles of intersection tend to zero. By compactness of the unit tangent bundle $T^1(S_h)$, we may pass to a subsequence of pairs to assume that

- the points of intersection converge in S_h, and
- the angles of intersections at these points go to zero.

Since laminations are closed subsets, we deduce that the laminations contain a common leaf, a contradiction to Proposition 10. □

Proof of Proposition (11.2). Suppose there is a sequence of pairs (ℓ_n^+, ℓ_n^-) of disjoint leaves in Λ_+ and Λ_- such that the distance between them tends to zero. By using cocompactness of the $\pi_1(S)$ action on $\widetilde{S}_h$, we may assume that the closest points between the leaves lie in a compact region in $\widetilde{S}_h$. Hence, we may pass to a convergent subsequence of pairs and further assume that the tangent vectors (to the leaves) at these pairs also converge. Since laminations are closed subsets, the sequence of leaves limit to a common leaf of Λ_+ and Λ_-, a contradiction by Proposition 10. □

Proof of Proposition (11.3). Suppose there is a sequence of leaves ℓ_n in Λ_+, containing subsegments $\sigma_n \subset \ell_n$ of length $|\sigma_n| = L_n \to \infty$ of Λ_+, such that the segments σ_n do not intersect Λ_-. By cocompactness, and the fact that laminations are closed, the subsegments σ_n limit to a leaf ℓ of Λ_+ which does not intersect Λ_-. This implies that there is a geodesic ray asymptotic to ℓ, which is disjoint from both laminations, contradicting the fact that the laminations bind the surface S_h. The exact same argument works with Λ_+ and Λ_- interchanged. □

2.2 Flow sets and ladders

The mapping torus construction determines a flow on the universal cover $\widetilde{S}_h \times \mathbb{R}$, i.e. a continuous 1-parameter family of homeomorphisms given by $F_t(p, s) = (p, s + t)$, which we will call the *suspension flow*. We shall consider the $\mathbb{R}$ component of $\widetilde{S}_h \times \mathbb{R}$ as "vertical".

Suppose that A is a subset of $\widetilde{S}_h$. We define the *suspension flow set* $F(A)$ to be

$$F(A) = \bigcup_{z\in\mathbb{R}} F_z(A).$$

If $A = \{p\}$ is a point, then $F(p)$ is just the suspension flow line through p. If A is a hyperbolic geodesic γ in $\widetilde{S}_h$, then the suspension flow set $F(\gamma)$ is called the *ladder* of γ. We will refer to γ as the *base* of the ladder.

2.3 The Cannon-Thurston metric

Let (S_h, Λ) be a hyperbolic surface together with a full pair of laminations, and let $\widetilde{S}_h$ be the universal cover of S_h. Following [CT07], we define an infinitesimal pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$ by

$$ds^2 = k^{2z}dx^2 + k^{-2z}dy^2 + (\log k)^2 dz^2. \tag{1}$$

Here log will mean the natural log base e and we will write $\log_k$ for log base k. Throughout this paper, we will use $k > 1$ exclusively to refer to the constant in the definition of the Cannon-Thurston metric. If the pair of laminations is the pair of invariant laminations determined by a pseudo-Anosov map f, we will choose $k > 1$ to be the stretch factor of f. With this choice, the monodromy map acts on the universal cover by vertical translation by one unit. For an arbitrary full pair of laminations, we may choose $k = e$.

The infinitesimal pseudo-metric gives rise to a pseudometric $d_{\widetilde{S}_h\times\mathbb{R}}$ on $\widetilde{S}_h \times \mathbb{R}$ in the standard manner:

- integrating the pseudometric along rectifiable paths gives a pseudo-distance; and then
- defining the distance between two points as the infimum of the length over all rectifiable paths connecting the two points.

The resulting pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$ is called the *Cannon–Thurston metric*. In the mapping torus case, the pseudo-metric is $\pi_1(M)$-invariant by construction. It is genuine pseudo-metric since if p and q are points in the same compact complementary region of $\widetilde{S}_h \setminus (\Lambda_+ \cup \Lambda_-)$, then for any $z_0 \in \mathbb{R}$, the corresponding points (p, z_0) and (q, z_0), with the same z-coordinate z_0, are distance zero apart in $\widetilde{S}_h \times \mathbb{R}$.

Theorem 12. [CT07, Theorem 5.1] *Let $f\colon S \to S$ be a pseudo-Anosov map, and let (S_h, Λ) be a hyperbolic metric on S together with a pair of invariant measured laminations for f, and let M be the corresponding fibered 3-manifold. Then the hyperbolic metric $d_{\mathbb{H}^3}$ and the $\pi_1(M)$–invariant global pseudometric $d_{\widetilde{S}_h\times\mathbb{R}} = \inf_\gamma \int_\gamma ds$ are quasi-isometric.*

If (S_h, Λ) is a full pair of laminations, then Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ is Gromov hyperbolic, as the vertical flow lines satisfy the flaring condition from the Bestvina-Feighn Combination Theorem [BF92].

Theorem 13. [BF92, page 88] *Let (S, Λ) be a hyperbolic metric on S together with a full pair of laminations. Then the Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ is Gromov hyperbolic.*

The Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ is quasi-isometric to $\mathbb{H}^3$ if and only if the pair of laminations have bounded geometry, by work of Rafi [Raf05]. In the bounded geometry case, the Cannon-Thurston metric is $\pi_1 S$-equivariantly quasi-isometric to $\mathbb{H}^3$, by the proof of the Ending Lamination Conjecture due to Minsky [Min10] and Brock, Canary and Minsky [BCM12].

The restriction of the infinitesimal pseudo-metric to the base fiber S_0 defines a pseudo-metric on $\widetilde{S}_h$. This metric is quasi-isometric to the hyperbolic metric $d_{\mathbb{H}^2}$, see Section 2.9 for further details.

Moving up in the z-direction expands distances in the x-direction and contracts them (by the reciprocal) in the y-direction. This is illustrated in Figure 1, where the horizontal lines are leaves of Λ_- and vertical lines are leaves of Λ_+.

The constant $\log k$ is chosen so that the map given by $(u, v) \mapsto (u, k^{-v})$ is an isometry from $\mathbb{R}^2$ with the metric $ds^2 = k^{2v}du^2 + (\log k)^2 dv^2$ to the upper half space model of $\mathbb{H}^2$ with the hyperbolic metric.

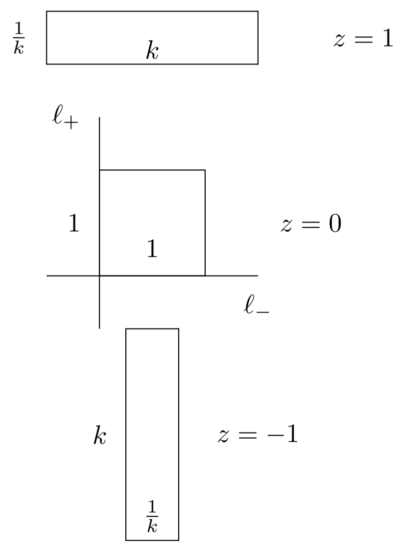

Figure 1: Rescaling arising from the vertical flow in the Cannon-Thurston metric.

Suppose that ℓ_+ is a leaf of Λ^+. The ladder $F(\ell_+)$ is then parametrized by the coordinates (y, z) in $\widetilde{S}_h \times \mathbb{R}$. By the definition of the pseudometric, $F(\ell_+)$ is a convex subset of $\widetilde{S}_h \times \mathbb{R}$. Moreover, $F(\ell_+)$ is quasi-isometric to $\mathbb{H}^2$, where in the upper half space model the quasi-isometry is given by $(y, z) \mapsto (y, k^{-z})$. The leaf ℓ_+ is a coarse horocycle, as is each image $F_z(\ell_+)$. The suspension flow lines are geodesics and the distance between two suspension flow lines decreases exponentially as the z-coordinate increases. Thus, as $z \to +\infty$, all suspension flow lines converge to the same limit point at infinity, as illustrated in Figure 2.

Similarly, if ℓ_- is a leaf of Λ_-, then the ladder $F(\ell_-)$ is parametrized by coordinates (x, z) in $\widetilde{S}_h \times \mathbb{R}$. The ladder $F(\ell_-)$ is again a convex subset of $\widetilde{S}_h \times \mathbb{R}$ quasi-isometric to $\mathbb{H}^2$, though in this case the quasi-isometry to the upper half space is given by $(x, z) \mapsto (x, k^z)$. The images of the leaf under the suspension flow, namely the $F_z(\ell_-)$, are coarse horocycles. The suspension flow lines are geodesics, and the distance between two suspension flow lines decreases exponentially as the z-coordinate decreases. Thus, as $z \to -\infty$, all suspension flow lines converge to the same limit point at infinity.

In fact, ladders over arbitrary geodesics in $\widetilde{S}_h$ are quasiconvex, a special case of a more general result of Mitra [Mit98]. We will state Mitra's result using the notation of this paper.

Theorem 14. [Mit98, Lemma 4.1] *Suppose that f is a pseudo-Anosov map and $\widetilde{S}_h$ is a hyperbolic metric. Then, there is a constant K, such that for any geodesic γ in $\widetilde{S}_h$, the ladder $F(\gamma)$ is K-quasiconvex in $\widetilde{S}_h \times \mathbb{R}$.*

Ladders over arbitrary geodesics are also quasi-isometric to $\mathbb{H}^2$, though we do not use this fact directly.

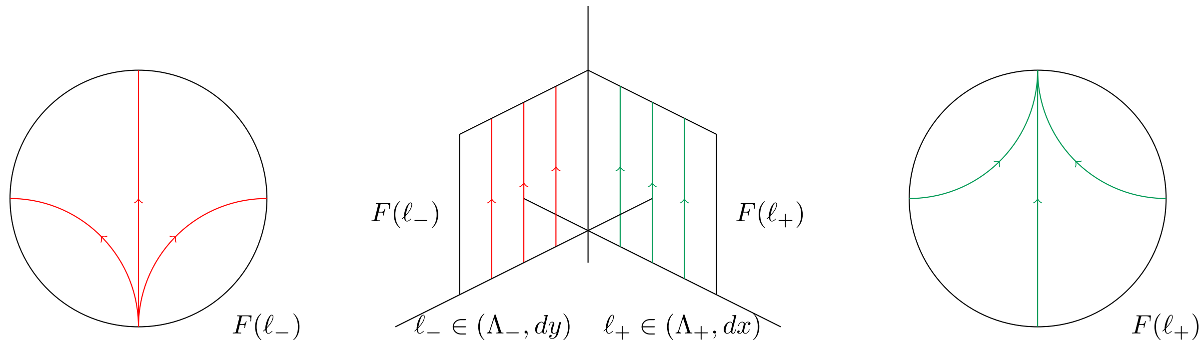

Figure 2: Ladders over leaves are quasi-isometric to $\mathbb{H}^2$.

2.4 Separation for ladders

Given two leaves ℓ and ℓ' of an invariant lamination, we define the distance between their ladders to be

$$d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell),F(\ell')) = \inf\left\{d_{\widetilde{S}_h\times\mathbb{R}}(p,p') \mid p\in F(\ell), p'\in F(\ell')\right\}.$$

In this section, we show that there is an $\epsilon=\epsilon_\Lambda>0$ such that for any pair of ladders, the Cannon–Thurston distance between them is either zero, or at least ϵ. Furthermore, we prove that the limit sets in $S^2_\infty=\partial\mathbb{H}^3$ of any two ladders $F(\ell)$ and $F(\ell')$ intersect if and only if they are distance zero apart.

As the first step, we show that the distance is zero with the infimum realized if and only if the base of the ladders is a pair of leaves that are boundary leaves of a common ideal complementary region.

Proposition 15. *Suppose that (S_h,Λ) is a hyperbolic structure on S together with a full pair of measured laminations. Suppose that ℓ and ℓ' are leaves of the same lamination. Then there are points $p\in F(\ell)$ and $p'\in F(\ell')$ with $d_{\widetilde{S}_h\times\mathbb{R}}(p,p')=0$ if and only if ℓ and ℓ' are boundary leaves of an ideal complementary region.*

Proof. Suppose that ℓ and ℓ' are boundary leaves of an ideal complementary region. Then we can find a subarc in $\widetilde{S}_h$ that connects ℓ and ℓ' such that its interior is disjoint from both laminations. The subarc then has length zero in the pseudo-metric, and so the distance between the ladders $F(\ell)$ and $F(\ell')$ is zero.

Conversely, suppose there is a path in $\widetilde{S}_h\times\mathbb{R}$ between $p\in F(\ell)$ and $p'\in F(\ell')$ that has zero length in the pseudo-metric. By the definition of the pseudo-metric, this path lies in a fiber $\widetilde{S}_h\times\{z\}$ and its interior is disjoint from the images (by the vertical flow F_z) of the laminations. This implies that ℓ and ℓ' are boundary leaves of a common ideal complementary region. □

For the remainder of this section, we set up some terminology. Suppose that ℓ and ℓ' are leaves of an invariant lamination. We denote the open strip in $\widetilde{S}_h$ with two boundary components ℓ and ℓ' by R. We denote by $A(R)$ the collection of arcs in $\widetilde{S}_h$ that have one endpoint on ℓ, the other endpoint on ℓ', and interior in R. The limit set of R is a disjoint union of two intervals, possibly with one of them a single point. We call these intervals I and I'. Let $\mathrm{Sep}_+(\ell,\ell')$ be the set of leaves in Λ_+ that separate ℓ from ℓ', that is, each leaf in $\mathrm{Sep}_+(\ell,\ell')$ is a leaf of Λ_+ that has one limit point in I, the other limit point in I'. Similarly, let $\mathrm{Sep}_-(\ell,\ell')$ of leaves in Λ_- that separate ℓ from ℓ'.

Lemma 16. *Suppose that ℓ and ℓ' are distinct leaves in Λ_+. Then $Sep_+(\ell,\ell')$ is non-empty if and only if ℓ and ℓ' are not boundary leaves of a common ideal complementary region of Λ_+.*

Proof. If $\mathrm{Sep}_+(\ell,\ell')$ is non-empty then ℓ and ℓ' cannot be boundary leaves of a common ideal complementary region of Λ_+.

Conversely, suppose that ℓ and ℓ' are not boundary leaves of a common ideal complementary region. Recall that I and I' are the two limit sets with one endpoint in ℓ, and the other endpoint in ℓ'. As the two geodesics ℓ and ℓ' are distinct, at least one of I and I' has non-empty interior.

If the interval I consists of a single point, its convex hull C_I is equal to I. If the interval I has non-empty interior, then the convex hull $C_I \subset R$ consists of the limit set I, together with all bi-infinite geodesics with both endpoints in I. In particular, the boundary of C_I is the bi-infinite geodesic α connecting the endpoints of I. Similarly, we denote by $C_{I'}$ the convex hull of I'. Again, if I' has non-empty interior, we denote by α' the bi-infinite geodesic connecting the endpoints of I'.

By convexity, any ideal complementary region of Λ_+ with all its ideal vertices in I is contained in C_I. Similarly, any ideal complementary region of Λ_+ with all its ideal vertices in I' is contained in $C_{I'}$.

The geodesics ℓ and ℓ', together with the geodesics α and α', form an ideal quadrilateral $T \subseteq R$ with at least three limit points, and so T has non-empty interior. Therefore, T must intersect an ideal complementary region U of Λ_+ contained in R. By construction, this complementary region has ideal vertices in both I and I'. We then find exactly two boundary leaves ℓ_1 and ℓ_2 of U connecting I to I'. If, as unordered pairs $(\ell_1, \ell_2) = (\ell, \ell')$, then ℓ and ℓ' are boundary leaves of a single ideal complementary region of Λ_+, a contradiction. We deduce that at least one of ℓ_1 or ℓ_2 is contained in $\text{Sep}_+(\ell, \ell')$, and thus $\text{Sep}_+(\ell, \ell')$ is non-empty, as required. □

By switching the invariant laminations, Lemma 16 also holds for Λ_- with $\text{Sep}_-(\ell, \ell')$ non-empty if and only if ℓ and ℓ' are not boundary leaves of a common ideal complementary region of Λ_-.

Lemma 17. *Suppose that ℓ and ℓ' are leaves of an invariant lamination. Then the subset $Sep_+(\ell, \ell')$ is non-empty if and only if the dx-measure of $Sep_+(\ell, \ell')$ is positive. Similarly, $Sep_-(\ell, \ell')$ is non-empty if and only if the dy-measure of $Sep_-(\ell, \ell')$ is positive.*

Proof. If the dx-measure of $\text{Sep}_+(\ell, \ell') > 0$ then $\text{Sep}_+(\ell, \ell')$ is non-empty by the definition of transverse measure. Conversely, suppose that $\text{Sep}_+(\ell, \ell')$ is non-empty and so there is a transverse arc contained in $A(R)$ that intersects in an interior point a leaf in $\text{Sep}_+(\ell, \ell')$. Since an invariant lamination has no isolated leaves, it follows that such an arc has positive dx-measure. Since the dx-measure of the arc is a lower bound on the dx-measure of $\text{Sep}_+(\ell, \ell')$, the lemma follows. □

Lemma 18. *Suppose that ℓ and ℓ' are leaves of an invariant lamination. Then $d_{\widetilde{S}_h \times \mathbb{R}}(F(\ell), F(\ell')) > 0$ if and only if the dx-measure of $Sep_+(\ell, \ell')$ and the dy-measure of $Sep_-(\ell, \ell')$ is positive.*

Proof. We let a be the dx-measure of $\text{Sep}_+(\ell, \ell')$ and b the dy-measure of $\text{Sep}_-(\ell, \ell')$. Suppose that γ is an arc in $A(R)$. Then γ must intersect in its interior every leaf in $\text{Sep}_+(\ell, \ell')$ and every leaf in $\text{Sep}_-(\ell, \ell')$. It follows that $dx(\gamma) \geqslant a$ and $dy(\gamma) \geqslant b$.

Suppose p and p' are points on ℓ and ℓ' respectively. By definition of the Cannon-Thurston metric, one of the two distances $d_{\widetilde{S}_h \times \mathbb{R}}((p, z), (p', z))$ or $d_{\widetilde{S}_h \times \mathbb{R}}((p, z'), (p', z'))$ is at most the distance $d_{\widetilde{S}_h \times \mathbb{R}}((p, z), (p', z'))$. Thus, we may assume that $z = z'$, that is, the pair of points are at the same height. Suppose that γ is an arc in $A(R)$ between p and p'. Then the length of $F_z(\gamma)$ equals $k^z dx(\gamma) + k^{-z} dy(\gamma) \geqslant k^z a + k^{-z} b$. The lemma now follows. □

We now show that the distance is zero but the infimum is not attained if and only if the ladders are over leaves that intersect a complementary region of the other lamination.

Proposition 19. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Suppose that ℓ and ℓ' are leaves of the same lamination and suppose that $d_{\widetilde{S}_h \times \mathbb{R}}(p, p') > 0$ for any pair of points $p \in F(\ell)$ and $p' \in F(\ell')$. Then the distance $d_{\widetilde{S}_h \times \mathbb{R}}(F(\ell), F(\ell')) = 0$ if and only if ℓ and ℓ' are not boundary leaves of a common ideal complementary region but intersect a common ideal complementary region of the other lamination.*

Proof. Breaking symmetry, we may assume that ℓ and ℓ' are leaves of Λ_+. The same argument holds for Λ_- by switching the laminations.

Suppose that $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) = 0$ but $d_{\widetilde{S}_h\times\mathbb{R}}(p, p') > 0$ for any pair of points $p \in F(\ell)$ and $p' \in F(\ell')$. By Proposition 15, ℓ and ℓ' are not boundary leaves of a common ideal complementary region of Λ_+. By Lemma 16, $\mathrm{Sep}_+(\ell, \ell')$ is non-empty. If $\mathrm{Sep}_-(\ell, \ell')$ is also non-empty, then by Lemma 18, $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) > 0$, a contradiction. Thus $\mathrm{Sep}_-(\ell, \ell')$ is empty. This means that there is an arc γ in $A(R)$ with endpoints p on ℓ and p' on ℓ' such that the interior of γ intersects only Λ_+. But then γ is contained in a single ideal complementary region of Λ_-, as required.

Conversely, suppose that ℓ and ℓ' are not boundary leaves of a single ideal complementary region of Λ_+ but intersect a common ideal complementary region of Λ_-. By Lemma 16, $\mathrm{Sep}_+(\ell, \ell')$ is non-empty. Since any arc γ in $A(R)$ intersects $\mathrm{Sep}_+(\ell, \ell')$, the length of $F_z(\gamma)$ is at least k^z times the dx-measure of $\mathrm{Sep}_+(\ell, \ell')$. In particular, this implies that $d_{\widetilde{S}_h\times\mathbb{R}}(p, p') > 0$ for any pair of points $p \in F(\ell)$ and $p' \in F(\ell')$. On the other hand, since ℓ and ℓ' intersect a common ideal complementary region of Λ_-, there is an arc γ in $A(R)$ with endpoints p on ℓ and p' on ℓ' such that the interior of γ intersects only Λ_+. Then the length of $F_z(\gamma)$ equals $k^z dx(\gamma)$ which goes to zero as $z \to -\infty$. Thus, $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) = 0$, as required. □

Finally, we show the distance gap for ladders.

Proposition 20. *Suppose that (S_h, Λ) is a hyperbolic structure on S together with a full pair of measured laminations. Then there is a constant $\epsilon_3 > 0$ such that for any two leaves ℓ_1 and ℓ_2 of a lamination, either $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_1), F(\ell_2)) \geqslant \epsilon_3$, or else $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_1), F(\ell_2)) = 0$.*

Proof. Suppose that there is a sequence of pairs of leaves ℓ_n and ℓ'_n of a lamination such that $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_n), F(\ell'_n)) > 0$ and tends to zero as $n \to \infty$. Breaking symmetry, we assume that ℓ_n and ℓ'_n are leaves of Λ_+. Since Λ_+ is a closed set, we may, by passing to a subsequence, assume that ℓ_n and ℓ'_n converge to leaves ℓ and ℓ' respectively. It follows that $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) = 0$.

By Proposition 15 and Proposition 19, we can find an arc α in $A(R)$ with endpoints p on ℓ and p' on ℓ' such that either the interior of α is disjoint from both laminations, or the interior of α intersects only Λ_+.

Suppose that q_n on ℓ_n and q'_n on ℓ'_n are sequences of points that converge to p and p'. It follows that by choosing q_n and q'_n sufficiently close to p and p' we can find an arc α_n with endpoints q_n and q'_n such that the interior of α_n intersects only Λ_+. But then by Proposition 19, $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_n), F(\ell'_n)) = 0$, a contradiction. □

Finally, we show that the limit sets of two ladders intersect if and only if they are distance zero apart.

Proposition 21. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. For any leaves ℓ and ℓ' in a lamination, $\overline{F(\ell)} \cap \overline{F(\ell')} \neq \varnothing$ if and only if $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) = 0$.*

Proof. Suppose that ℓ and ℓ' are in Λ_+ and $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) = 0$. By Proposition 15 and Proposition 19, ℓ and ℓ' are either boundary leaves of an ideal complementary region of Λ_+, or else intersect an ideal complementary region of Λ_-. It follows that there are points $p \in \ell$ and $p' \in \ell'$ and an arc α in $A(R)$ with endpoints p and p' such that the interior of α is either disjoint from both laminations or intersects only Λ_+.

If the interior is disjoint from both laminations then $F_z(p)$ and $F_z(p')$ are pseudo-metric distance zero for all z. Thus, the flow lines determine the same limit points at infinity.

Now suppose that the interior of α intersects Λ_+. Then $dx(\alpha) > 0$ and $dy(\alpha) = 0$. Then the distance between $F_z(p)$ and $F_z(p')$ is $k^z dx(\alpha)$ and hence the flow lines determine the same point at infinity as $z \to -\infty$.

Conversely, suppose $\overline{F(\ell)} \cap \overline{F(\ell')}$ is non-empty and let z_∞ be a point of the intersection. Every limit point in $\overline{F(\ell)}$ (similarly $\overline{F(\ell')}$) is a limit point of a suspension flow line, and hence there are points $p \in \ell$ and $p' \in \ell'$ such that their suspension flow lines $F(p)$ and $F(p')$ converge to z_∞ in one direction. The Cannon-Thurston metric is δ-hyperbolic and suspension flow lines are geodesics. We deduce that $F(p)$ and $F(p')$ are bounded distance in the direction of the common limit point. It follows that there is an arc β in $A(R)$ with endpoints p and p' such that the interior of β intersects only one of the laminations. Thus, the distance between $F(\ell)$ and $F(\ell')$ is zero, as desired. □

2.5 Quasigeodesics

We recall some basic facts about quasigeodesics, see for example Bridson and Haefliger [BH99, III.H].

Definition 22. Let (X, d) be a geodesic metric space and let $\gamma\colon I \to X$ be a path, where I is a (possibly infinite) connected subset of $\mathbb{R}$. Let $Q \geqslant 1$ and $c \geqslant 0$ be constants. The path γ is a *(Q, c)-quasigeodesic* if for all t_1 and t_2 in I,

$$\frac{1}{Q}|t_2 - t_1| - c \leqslant d(\gamma(t_1), \gamma(t_2)) \leqslant Q|t_2 - t_1| + c.$$

By [BH99, III.H Lemma 1.11], given a (Q, c)-quasigeodesic, there is a continuous (Q, c')-quasigeodesic with the same endpoints, so for our purposes we may assume that all quasigeodesics are continuous, and we will do so from now on.

A *reparametrization* of a path $\gamma\colon \mathbb{R} \to X$ is the path $\gamma \circ \rho$, where $\rho\colon \mathbb{R} \to \mathbb{R}$ is a proper non-decreasing function. We say a path $\gamma\colon \mathbb{R} \to X$ is an *unparametrized* (Q, c)-quasigeodesic if there is a reparametrization of γ which is a (Q, c)-quasigeodesic.

We will use the following stability property for quasigeodesics in hyperbolic spaces, known as the Morse Lemma.

Lemma 23. [BH99, Theorem 1.7] *Let X be a δ-hyperbolic space. Then for any Q and c there is a constant L such that any (Q, c)-quasigeodesic is contained in an L-neighborhood of the geodesic connecting its endpoints.*

2.6 Nearest point projections and fellow traveling

Suppose that α is a subset of a Gromov hyperbolic space (X, d). The *nearest point projection* $p_\alpha\colon X \to \alpha$ sends each point $x \in X$ to a closest point to x in α. If α is Q-quasiconvex, then the nearest point projection is K-coarsely well defined, where K depends only on Q and the constant δ of hyperbolicity.

Suppose that α and β are two geodesics in X. We define the *K-fellow traveling set* for α with respect to β to be the subset of α contained in a K-neighborhood of β, i.e. $\alpha \cap N_K(\beta)$. If the diameter of the projection image $p_\alpha(\beta)$ is sufficiently large, then $p_\alpha(\beta)$ is contained in a bounded neighborhood of the geodesic β, and so is contained in a fellow traveling set.

In the special case that X is $\mathbb{H}^n$ and α is a geodesic, the closest point on α to any point $x \in X$ is unique, and so p_α is well-defined. Furthermore, for any geodesic β, the image $p_\alpha(\beta)$ is a subinterval of α, which we will refer to as the *nearest point projection interval*, or just the *projection interval*. Similarly, any K-fellow traveling set $\alpha \cap N_K(\beta)$ is also an interval, which we shall call the *K-fellow traveling interval*.

For $\mathbb{H}^n$, the hyperbolicity constant is $\delta = 2\log 3$. We shall write δ_2 for the hyperbolicity constant for the pseudometric $d_{\widetilde{S}_h}$ on $\widetilde{S}_h$, and δ_3 for the hyperbolicity constant for the pseudometric $d_{\widetilde{S}_h \times \mathbb{R}}$ on $\widetilde{S}_h \times \mathbb{R}$. Both constants depend on the pseudo-Anosov f.

A standard consequence of δ-hyperbolicity is the useful property stated below that if the projection image of a geodesic β onto another geodesic α is large, then the two geodesics fellow travel and the projection image $p_\alpha(\beta)$ is contained in a bounded neighborhood of β.

Proposition 24. [KS24, Lemma 1.120] *Suppose that X is a δ-hyperbolic space and suppose that α and β are geodesics in X. If the diameter of the projection image $p_\alpha(\beta)$ is greater than 8δ, then the projection image is contained in a 6δ-neighborhood of β, i.e. $p_\alpha(\beta) \subseteq N_{6\delta}(\beta)$, and so $p_\alpha(\beta)$ is contained in the fellow traveling set $\alpha \cap N_{6\delta}(\beta)$.*

In $\mathbb{H}^2$, if two geodesics intersect at angle θ, then the size of the projection interval is roughly $\log(1/\theta)$. In fact,the same result holds for two geodesics that do not intersect, but are distance θ apart. We will use these properties in Section 6 below.

Proposition 25. *There is a constant $T_0 \geqslant 0$, such that for any unit speed geodesic γ_1 in $\mathbb{H}^2$ and any geodesic γ_2 such that*

- *γ_2 intersects γ_1 at the point $\gamma_1(0)$ at an angle $0 < \theta \leqslant \pi/2$, or*
- *the distance from γ_1 to γ_2 is $\theta > 0$, and the closest point occurs at $\gamma_1(0)$,*

then the nearest point projection interval $p_{\gamma_1}(\gamma_2)$ is equal to $\gamma_1([-T, T])$, where

$$\log \frac{1}{\theta} \leqslant T \leqslant \log \frac{1}{\theta} + T_0,$$

and furthermore, for all $|t| \leqslant \log \frac{1}{\theta}$, the distance from $\gamma_1(t)$ to γ_2 is at most $3/2$.

This is well known, we provide the details in Appendix A for the convenience of the reader. In fact, for small $|t|$, the two geodesics are exponentially close, see Section 5.1 for further details.

We now record the useful fact that if two geodesics α and β intersect at angle θ, then the size of their projection intervals onto each other is roughly $\log \frac{1}{\theta}$, and furthermore, for any other geodesic γ, the overlap between the projection intervals for α and β on γ is bounded in terms of θ.

Proposition 26. *For any constant $\alpha_\Lambda > 0$ there is a constant $\rho_\Lambda > 0$ such that for any two geodesics in $\mathbb{H}^2$ which intersect at angle $\theta \geqslant \alpha_\Lambda$, and for any other geodesic γ, the intersection of the nearest point projection intervals of α and β to γ has diameter at most ρ_Λ.*

Proof. Suppose α and β intersect at the point p with angle $\theta \geqslant \alpha_\Lambda$. We shall choose $\rho_\Lambda = 2 \log \frac{1}{\alpha_\Lambda} + 4T_0 + 16$. By Proposition 25, the radius of the nearest point projection interval of α to β, and also of β to α, is at most $\log \frac{1}{\alpha_\Lambda} + T_0$.

For any geodesic γ, let $I_\alpha = p_\gamma(\alpha)$ and $I_\beta = p_\gamma(\beta)$ be the nearest point projection intervals of α and β onto γ. Suppose that their overlap has size at least ρ_Λ, i.e. the length of $I_\alpha \cap I_\beta$ is at least ρ_Λ. If we truncate I_α and I_β by length T_0 at both ends, then the truncated intervals have overlap of length at least $\rho_\Lambda - 2T_0$. Let I be the interval of overlap for the truncated projection intervals, i.e. I is the closure of $I_\alpha \cap I_\beta \setminus N_{T_0}(\partial(I_\alpha \cap I_\beta))$.

By Proposition 25, each endpoint of I is distance at most $3/2$ from both α and β. We denote by a_1 and a_2 the points on α closest to each endpoint of I. It follows that the distance between a_1 and a_2 is at least $|I| - 2T_0 - 3$, and each point a_i is distance at most 3 from β.

We pick points b_i in β distance at most 3 from the points a_i. Then the nearest point projection of b_i to α is distance at most 6 from a_i. In particular, the diameter of the nearest point projection interval of β to α is at least $|I| - 2T_0 - 15 \leqslant \rho_\Lambda - 2T_0 - 15$. It follows from our choice of ρ_Λ that the diameter of the projection interval of β onto α is at least $2 \log \frac{1}{\alpha_\Lambda} + 2T_0 + 1$, a contradiction. □

Finally, we show that if two geodesics α and β have strictly nested projection intervals onto a third geodesic γ, i.e. $p_\gamma(\alpha) \subset p_\gamma(\beta)$, and α intersects γ, then α and β also intersect.

Proposition 27. *Let γ be a geodesic in $\mathbb{H}^2$ which intersects a geodesic ℓ_1, with projection interval $I_1 \subset \gamma$. Let ℓ_2 be a geodesic with projection interval $I_2 \subset \gamma$, such that $I_1 \subset I_2$. Then ℓ_1 and ℓ_2 intersect.*

Proof. Consider the nearest point projection map $p \colon \mathbb{H}^2 \to \gamma$. Consider the complement of the pre-image of I_1, i.e. $\mathbb{H}^2 \setminus p^{-1}(I_1)$. This has two connected components, which are separated by the geodesic ℓ_1. As I_1 is a strict subset of I_2, each endpoint of ℓ_2 is contained in a different complementary component, and so the endpoints of ℓ_2 are separated by ℓ_1, and so the two geodesics ℓ_1 and ℓ_2 intersect. □

2.7 Lebesgue measures and the geodesic flow

We review the properties of the geodesic flow we will use, see for example [EW11].

A choice of basepoint x_0 in $\mathbb{H}^n$ determines a measure on the boundary sphere $\partial\mathbb{H}^n$, for example by choosing the disc or ball model for $\mathbb{H}^n$ with the basepoint as the center point, and giving $\partial\mathbb{H}^n$ the measure induced from the standard metric on the unit sphere. We will call this measure *Lebesgue measure*, and this measure depends on the choice of basepoint, though we will suppress this from our notation. Different choices of basepoint give measures which are absolutely continuous with respect to each other.

Let Γ be a discrete cocompact subgroup of isometries of $\mathbb{H}^n$, and let $M = \mathbb{H}^n/\Gamma$. Hopf [Hop71] showed that the geodesic flow g_t on the unit tangent bundle $T^1(M)$ is ergodic with respect to Liouville measure. As M is a compact hyperbolic manifold of constant negative curvature, Liouville measure is proportional to the Bowen-Margulis-Sullivan measure, the measure of maximal entropy for the geodesic flow. Liouville measure on $T^1(M)$ is the product of the measure determined by the hyperbolic metric on M, with Lebesgue measure on the unit tangent spheres. As the conditional measures on each unit tangent sphere determined by Liouville measure are Lebesgue measures, the measure on geodesics in M induced by Liouville measure is absolutely continuous with respect to the measure on geodesics in M induced by the product of Lebesgue measures on $(\partial\mathbb{H}^n \times \partial\mathbb{H}^n) \setminus \Delta$.

For the case of the Lebesgue surface measure, $n = 2$ and Γ is the fundamental group of the surface, $\pi_1 S$. For the case of the Lebesgue 3-manifold measure, $n = 3$, and Γ is the fundamental group of the mapping torus, $\pi_1 M$. In both cases, we will use the fact that a geodesic chosen according to the product of Lebesgue measures on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ is uniformly distributed in the unit tangent bundle of the compact quotient manifold, almost surely. We will use the following version of this result.

Proposition 28. *Let Γ be a discrete cocompact group of isometries of $\mathbb{H}^n$, and let B be a Borel set in $M = \mathbb{H}^n/\Gamma$. Then for almost all geodesics γ in $T^1(M)$,*

$$\lim_{T\to\infty} \frac{1}{T} |\gamma([0,T]) \cap B| = \mathbf{vol}(B).$$

2.8 Random walks and hitting measures

We recall some results in the theory of random walks on countable groups acting on Gromov hyperbolic spaces. Suppose that a countable group G acts on a δ-hyperbolic space X. The results we state do not require X to be locally compact or the action to be locally finite. But for our purposes, $G \curvearrowright X$ will always be either $\pi_1(S) \curvearrowright \widetilde{S}_h$ or $\pi_1(M) \curvearrowright \widetilde{S}_h \times \mathbb{R}$, which are cocompact actions on locally compact spaces. We say the action of a group G on a space X is *nonelementary* if it contains two independent loxodromic isometries of X. We say a probability measure μ on G is *geometric* if it has finite support and the semigroup generated by its support is equal to G.

A random walk of length n on G is a random product $w_n = g_1 \cdots g_n$ where each g_i is chosen independently according to a probability measure μ on G. We call the elements g_i the *steps* of the random walk. Passing to infinitely many steps, we may consider the sequence of steps (g_n) to be an element of $(G,\mu)^{\mathbb{Z}}$. We call $(G,\mu)^{\mathbb{Z}}$ the *step space*. The *location* w_n at time n is determined by $w_0 = 1 \in G$, and $w_{n+1} = w_n g_{n+1}$ for all $n \in \mathbb{Z}$. The *location space* is the probability space $(G^{\mathbb{Z}}, \mathbb{P})$, where $\mathbb{P}$ is the pushforward of the product measure under the map that sends (g_n) to (w_n). A choice of basepoint $x_0 \in X$ gives a sequence $(w_n x_0)$ which we shall also call a *sample path* of the random walk.

Theorem 29. [Kai94][MT18] *Suppose that G is a countable group that has a nonelementary action on a Gromov hyperbolic space X. Suppose that μ is a nonelementary probability measure on G, that is, the support of μ generates a nonelementary subgroup of G for its action on X. Then almost all (bi-infinite) sample paths $w = (w_n)_{n\in\mathbb{Z}}$ for the μ-random walk on G, the sequences $(w_n x_0)$ converge as $n \to \infty$ and $n \to -\infty$ to the Gromov boundary ∂X of X. The convergence defines a pair of non-atomic measures ν and $\check{\nu}$ on ∂X.*

We call the measures on ∂X obtained in Theorem 29 the *hitting measures* for the random walk. If μ is symmetric, then the forward and backward hitting measures are equal, i.e. $\nu = \check{\nu}$.

For almost every sample path $w = (w_n)_{n\in\mathbb{Z}}$, the forward and backward limits $x_\infty^+ = \lim_{n\to\infty} w_n x_0$ and $x_\infty^- = \lim_{n\to-\infty} w_n x_0$ are different from each other. Hence, almost every bi-infinite sample path w defines a bi-infinite geodesic γ_w in X and all choices for γ uniformly fellow travel, in the sense that the fellow traveling constant depends only on the Gromov-hyperbolicity constant. Making a choice for the geodesic, we call it the geodesic *tracked* by the sample path.

The distance $d_X(x_0, w_n x_0)$ grows linearly with high probability. This was shown by [BMSS23] for μ with finite exponential moment, and by Gouëzel [Gou22, Theorem 1.1] for general μ.

Lemma 30. [BMSS23][Gou22] *Suppose that G is a countable group with a nonelementary action on a Gromov hyperbolic space X. Suppose that μ is a nonelementary probability measure on G. Then there are constants $\ell > 0, K \geqslant 0$ and $0 < c < 1$ such that*

$$\mathbb{P}(d_X(x_0, w_n x_0) \leqslant \ell n) \leqslant K c^n.$$

Furthermore, if μ has finite exponential moment in X, then for any $\epsilon > 0$ there are constants $\ell > 0, K \geqslant 0$ and $c < 1$ such that

$$\mathbb{P}(|\tfrac{1}{n} d_X(x_0, w_n x_0) - \ell| \geqslant \epsilon) \leqslant K c^n.$$

The Gromov product of three points a, b, c in a metric space X is defined to be

$$(b \,.\, c)_a = \tfrac{1}{2}(d_X(a,b) + d_X(a,c) - d_X(b,c))$$

When X is δ-hyperbolic, the Gromov product $(b \,.\, c)_a$ equals the distance from a to a geodesic from b to c, up to a bounded error depending only on δ. In this case, the product can be extended to points which lie in the boundary, i.e. we may choose $b, c \in \overline{X} = X \cup \partial X$.

The "shadow" will, roughly speaking, be the set of all points c in $\overline{X}$ such that any geodesic from a to c passes close to b. Formally:

Definition 31. Suppose that a is a point in a Gromov hyperbolic space X and b is a point in $\overline{X}$. Suppose that $r \geqslant 0$ is a constant. The *shadow* $\mho_a(b, r) \subseteq \overline{X}$ consists of the closure of the set of all points c, such that $(b \,.\, c)_a \geqslant r$.

This differs from the usual definition of shadows, which is the closure of the following set, where again $a \in X$ and $b \in \overline{X}$:

$$\mho_a(b, r) = \{c \in \overline{X} \mid (b \,.\, c)_a \geqslant d_X(a, b) - r\}.$$

The two definitions are equivalent by setting $r = d_X(a, b) - R$ for points $b \in X$, but our definition extends more conveniently to points $b \in \partial X$.

We will use the following Gromov product estimates from [BMSS23]. We state the version incorporating the linear progress results of [Gou22]. As shadows are defined in terms of the Gromov product, we also state the results in terms of shadows.

Proposition 32. [BMSS23, Proposition 2.11][Gou22, Theorem 1.1] *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov-hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G with a finite exponential moment. Then there are constants $K > 0$ and $c < 1$ such that for all $0 \leqslant i \leqslant n$ and all $R > 0$ one has*

$$\mathbb{P}\left((x_0 \,.\, w_n x_0)_{w_i x_0} \geqslant R\right) \leqslant K c^R,$$

and in particular

$$\mathbb{P}\left(d_X(w_n x_0, \gamma_w) \geqslant R\right) \leqslant K c^R.$$

Furthermore, using the definition of shadows,

$$\mathbb{P}\left(\nu(\mho_{x_0}(w_n x_0, R))\right) \leqslant K c^R.$$

The Borel-Cantelli Lemma then gives the following corollary.

Corollary 33. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G with finite exponential moment with respect to X. Then there is a constant $D > 0$, such that for almost all bi-infinite sample paths $w = (w_n)$, there is a tracked geodesic $\gamma = \gamma_w$, and an integer N such that for all $n \geqslant N$*

$$d_X(w_n x_0, \gamma) \leqslant D \log n.$$

Proof. By Proposition 32, for any $D > 0$ we have

$$\mathbb{P}(d_X(w_n x_0, \gamma) \geqslant D \log n) \leqslant K \exp(D \log n \log c).$$

We choose D such that $D \log c < 1$. Set $\alpha = D \log c$. Then

$$\sum_n \mathbb{P}(d_X(w_n x_0, \gamma) \geqslant D \log n) \leqslant K \sum_n \frac{1}{n^\alpha} < \infty.$$

Hence, by the Borel-Cantelli Lemma, we deduce that for almost every sample path, there is a natural number N (that depends on the path) such that $d(w_n x_0, \gamma) < D \log n$ for all $n \geqslant N$. ☐

Let $\gamma(t_n)$ be a closest point on γ to $w_n x_0$. Then combining Lemma 30 and Proposition 32 gives the following linear progress result for the t_n.

Corollary 34. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G. Then there are constants $\ell > 0, K \geqslant 0$ and $0 < c < 1$ such that*

$$\mathbb{P}(t_n \leqslant \ell n) \leqslant K c^n.$$

Furthermore, if μ has finite exponential moment with respect to X, then for any $\epsilon > 0$ there are constants $\ell > 0, K \geqslant 0$ and $), c < 1$ such that

$$\mathbb{P}(|\tfrac{1}{n} t_n - \ell| \geqslant \epsilon) \leqslant K c^n.$$

We now use the Borel-Cantelli Lemma to show that the gap between t_n and t_{n+1} is at most $\log n$.

Proposition 35. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G with finite exponential moment with respect to X. Then there is a constant $D > 0$, such that for almost all sample paths, there is a tracked geodesic γ, and an integer N such that for all $n \geqslant N$*

$$|t_{n+1} - t_n| \leqslant D \log n.$$

Proof. By the finite exponential moment of μ in X, there are constants $K_1 > 0$ and $c_1 < 1$ such that $\mathbb{P}(d_X(w_n x_0, w_{n+1} x_0) \geqslant R) \leqslant K_1 c_1^R$. By Proposition 32, there are constants $K_2 > 0$ and $c_2 < 1$ such that for all n, we have that $\mathbb{P}(d_X(w_n x_0, \gamma) \geqslant R) \leqslant K_2 c_2^R$. By the triangle inequality,

$$|t_{n+1} - t_n| \leqslant d_X(\gamma(t_n), w_n x_0) + d_X(w_n x_0, w_{n+1} x_0) + d_X(w_{n+1} x_0, \gamma(t_{n+1})).$$

If $|t_{n+1} - t_n| \geqslant 3R$, then at least one of the terms on the right is at least R. Therefore, $\mathbb{P}(|t_{n+1} - t_n| \geqslant 3R) \leqslant \max\{K_1 c_1^R, K_2 c_2^R\}$. The result then follows from the Borel-Cantelli Lemma applied the same way as in the proof of Corollary 33. ☐

Finally, we verify that the $(\nu \times \check{\nu})$-measure, as defined in Theorem 29, of the diagonal $\Delta \subset \partial X \times \partial X$ is zero. We may define a neighborhood of the diagonal as follows (using any basepoint $a \in X$).

$$N_r(\Delta) = \bigcup_{b \in \partial X} \mho_a(b, r) \times \mho_a(b, r)$$

Lemma 36. [Mah12, Proposition 4.7] *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a geometric probability measure on G. Then there are constants $K \geqslant 0$ and $c < 1$ such that $\nu \times \check{\nu}(N_r(\Delta)) \leqslant Kc^r$, with $\nu \times \check{\nu}$ again as in Theorem 29.*

The above results imply that the distance from a location $w_n x_0$ to the tracked geodesic γ is given by a probability measure with exponential decay.

Proposition 37. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G. Then there are constants $K \geqslant 0$ and $c < 1$ such that $\mathbb{P}(d_X(w_n x_0, \gamma) \geqslant r) \leqslant Kc^r$.*

Proof. By the action of G on $(\partial X \times \partial X, \nu \times \check{\nu})$, it suffices to show this for $d_X(x_0, \gamma)$. Suppose that $d_X(x_0, \gamma) \geqslant r$. As the Gromov product of three points $(b \cdot c)_a$ is, up to an error of at most 2δ, equal to the distance from a to a geodesic from b to c, the endpoints of γ are contained in the neighborhood $N_{r+2\delta}(\Delta)$. The probability that this occurs is at most $\nu \times \check{\nu}(N_{r+2\delta}(\Delta)) \leqslant Kc^{r+2\delta}$, as required. □

2.9 The pseudo-metric on the surface and the flat metric

In this section we review some well known results that relate hyperbolic and flat metrics on surfaces, see for example [Kap01, Chapter 11].

A flat structure S_q on a closed surface S is a Euclidean cone metric with finitely many cone points, each of whose angles are integer multiples of π. Furthermore, for any sufficiently small coordinate chart disjoint from the cone points, there is a preferred choice of orthogonal directions, known as either the horizontal and vertical directions, or alternatively the real and imaginary directions. The integral lines of the horizontal directions give a (singular) foliation called the horizontal foliation. Similarly, the integral lines of the vertical directions give a (singular) foliation called the vertical foliation. Flat lengths of orthogonal arcs define transverse measures for the foliations.

The Cannon-Thurston metric given by Equation (1) is $\pi_1(S)$-invariant. Hence, the restriction of the infinitesimal pseudometric on $\widetilde{S}_h \times \mathbb{R}$ to S_0 gives rise to an infinitesimal pseudometric on S_h, which we have denoted by $d_{\widetilde{S}_h}$. By identifying points on S_h that are pseudo-metric distance zero apart, i.e. points which lie in the same component of $S_h \setminus (\Lambda_+ \cup \Lambda_-)$, we get a quotient of S_h which is isometric to a flat metric S_q on S. The quotient map sends the invariant laminations (Λ_-, dy) and (Λ_+, dx) to the real and imaginary measured foliations ($\mathcal{F}_r$ and $\mathcal{F}_i$ respectively) for the flat metric S_q. As both (pseudo-)metrics are defined on the closed surface S, they give quasi-isometric metrics on the universal cover. We record this statement as a proposition to fix notation.

Proposition 38. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Then the hyperbolic metric $d_{\mathbb{H}^2}$ and the Cannon-Thurston pseudometric $d_{\widetilde{S}_h}$ on the universal cover $\widetilde{S}_h$ are quasi-isometric, i.e. there are constants $Q_\Lambda \geqslant 1$ and $c_\Lambda \geqslant 0$ such that for any points p and p' in the universal cover,*

$$\frac{1}{Q_\Lambda} d_{\widetilde{S}_h}(p, p') - c_\Lambda \leqslant d_{\mathbb{H}^2}(p, p') \leqslant Q_\Lambda d_{\widetilde{S}_h}(p, p') + c_\Lambda.$$

By compactness of the circle direction or equivalently $\mathbb{Z}$-periodicity by the action of the pseudo-Anosov f, the constants in Proposition 38 remain uniform over any choice S_z as fiber. More generally, if a doubly degenerate surface group in $\mathrm{PSL}(2, \mathbb{C})$ has bounded geometry then the constants in Proposition 38 will remain uniform for the quasi-isometry between the pseudo-metric and the flat metric for any S_z.

2.10 Quasigeodesics in the singular solv metric

In this section, we discuss for context McMullen's construction of quasigeodesics using saddle connections in the flat metric [McM01] and how our work relates to it. The rest of the paper does not depend on the material in this section.

Solv geometry on $\mathbb{R}^3$ is the metric on $\mathbb{R}^3$ given by the infinitesimal metric $ds^2 = k^{2z}dx^2 + k^{-2z}dy^2 + dz^2$, where $dx^2 + dy^2$ is the standard Euclidean metric in $\mathbb{R}^2$. A cone metric modeled on solv geometry is a metric on $\mathbb{R}^3$ in which each point has a neighborhood isometric to either a neighborhood in solv geometry, or a neighborhood in a branched cover of solv geometry, where the branch set is the vertical z-axis. By identifying points of $\widetilde{S}_h \times \mathbb{R}$ that are distance zero apart in the Cannon-Thurston pseudo-metric, we obtain a genuine metric on $\widetilde{S}_h \times \mathbb{R}$. The resulting metric, called the *singular solv metric*, is a cone metric modeled on solv geometry. It is quasi-isometric to the Cannon-Thurston metric; in fact, this holds for any doubly degenerate surface group in $\mathrm{PSL}(2, \mathbb{C})$ with bounded geometry. The cone singularities are the suspension flow lines through the cone points of the flat metric on S_0, see McMullen [McM01] or Hoffoss [Hof07] for further details.

McMullen [McM01] gives an explicit construction of quasigeodesics in the singular solv cone metric on $\widetilde{S}_q \times \mathbb{R}$. We briefly describe this construction. A *saddle connection* in the universal cover $\widetilde{S}_q$ is a flat geodesic between two cone points with no cone points in its interior. A "typical" (in the sense defined in [CP25b]) bi-infinite flat geodesic in $\widetilde{S}_q$ consists of a concatenation of saddle connections s_i such that

- the end cone point of s_i is the beginning cone point for s_{i+1}, and
- the angles between s_i and s_{i+1} (clockwise and counter-clockwise) are both at least π.

The restricted metric on each fiber $\widetilde{S}_q \times \{z\}$ is given by the original flat metric $\widetilde{S}_q$ scaled by k^z in the x-direction, and k^{-z} in the y-direction. For each $z \in \mathbb{R}$, one can consider the length of $F_z(s_i)$ in the flat metric on $\widetilde{S}_q \times \{z\}$. The *optimal height* for a saddle connection s_i is the value of z for which this flat length is minimized. Suppose that a saddle connection has slope m_i in $\widetilde{S}_q$. An elementary calculation shows that the optimal height for s_i is $\log m_i$. We may construct a path in $\widetilde{S}_q \times \mathbb{R}$ by

- placing each saddle connection s_i at its optimal height, and
- connecting the endpoints of adjacent saddle connections by suspension flow segments.

We call this an *optimal height path*.

Theorem 39. [McM01]*Suppose that S is an orientable surface of finite type, that is with negative Euler characteristic and finite area. Suppose that f is a pseudo-Anosov map of S. Then there are constants Q and c such that for any bi-infinite geodesic in $\widetilde{S}_q$, the corresponding optimal height path in the singular solv metric on $\widetilde{S}_q \times \mathbb{R}$ is (Q, c)-quasigeodesic.*

We do not know how to deduce our results (specifically Theorem 7) directly from optimal height quasigeodesics in the singular solv metric. There are two issues:

- Quasigeodesics in the singular solv metric on $\widetilde{S}_q \times \mathbb{R}$ determine quasigeodesics in the Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$, but we do not know how to obtain precise enough information to use the ergodicity of the geodesic flow on S_h, which underpins our averaging argument. We remark that it might be possible to apply recent work of Cantrell and Pollicott [CP25a], which gives statistics for the relation between the geometric length along geodesics and the number of saddle connections. However, we make no attempt to pursue this direction here.
- The extension of the quasi-isometry between the hyperbolic and the flat metric on $\widetilde{S}_h$ to the boundary circle is likely to be singular, that is, the pushforward of the Lebesgue measure becomes singular.

Instead, we construct (in Section 6) explicit quasi-geodesics in the Cannon-Thurston metric from which we can deduce precise enough information. We then use the hyperbolic metrics on the surface and the 3-manifold for the averaging process using the ergodicity of the geodesic flow.

3 Singularity of measures

In this section, we prove the singularity of measures, namely Theorem 4, by showing that typical geodesics have different behavior for the pushforwards of the surface measures (Theorem 6) than for the 3-manifold measures (Theorem 8). We show that for almost all geodesics in $\widetilde{S}_h$ with respect to the surface measures, the geodesics they determine in $\widetilde{S}_h \times \mathbb{R}$ spend a positive proportion of time close to the base fiber. On the other hand, for almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$ with respect to the 3-manifold measures, the proportion of time spent close to the base fiber tends to zero. We prove these facts for Lebesgue measures in Section 3.2 and for hitting measures for random walks in Section 3.3. We start by recording some useful facts about geodesics which we will use in the subsequent sections.

Suppose that γ is an oriented geodesic in $\widetilde{S}_h$. We denote its pair of limit points in $\partial\widetilde{S}_h$ by γ_+ and γ_-.

Definition 40. We say that a bi-infinite geodesic γ in $\widetilde{S}_h$ is *non-exceptional* if

- its limit points γ_- and γ_+ are distinct from the limit points of any boundary leaf of any ideal complementary region of either of the invariant laminations, and
- the limit points have distinct images under the Cannon-Thurston map, that is $\iota(\gamma_-) \neq \iota(\gamma_+)$.

As we see below, for surface measures, almost all geodesics in $\widetilde{S}_h$ are non-exceptional.

Proposition 41. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Let ν be one of the surface measures from Definition 1 or Definition 2. Then ν-almost all geodesics in $\widetilde{S}_h$ are non-exceptional.*

Proof. Let γ be a geodesic in $\widetilde{S}_h$. Suppose that its image $\iota(\gamma)$ has a single limit point at infinity. Then γ is either a leaf of an invariant lamination, or contained in an ideal complementary region. Being ideal polygons with finitely many sides, there are only countably many ideal complementary regions. So the collection of geodesics contained in ideal complementary regions has measure zero as the measures are all non-atomic.

So we may consider leaves of invariant laminations. For Lebesgue measure, Birman and Series [BS85, Theorem II] showed that the collection of endpoints of all simple geodesics has measure zero in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. For hitting measure, this result follows from double ergodicity of the action of $\pi_1(S)$ on the boundary, due to Kaimanovich [Kai03, Theorem 17]. For completeness, we give the details for hitting measure below.

Let Δ denote the diagonal in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Suppose Λ is a geodesic lamination and $\partial\Lambda \subset \partial\widetilde{S}_h \times \partial\widetilde{S}_h \setminus \Delta$ be the endpoints of leaves of Λ. The action of the fundamental group $\pi_1(S)$ on $\partial\widetilde{S}_h \times \partial\widetilde{S}_h \setminus \Delta$ is ergodic for the hitting measure. Since Λ is $\pi_1(S)$-invariant, the set $\partial\Lambda$ has measure 0 or 1. Suppose ℓ is a leaf of Λ. Suppose that γ is a geodesic that crosses ℓ. Then γ does not lie in Λ. We can then choose small neighborhoods U_+ and U_- of γ_+ and γ_- such that any geodesic with one endpoint in U_+ and the other endpoint in U_- also crosses ℓ. Thus, such a geodesic does not lie in Λ. As open sets have positive measure, $\partial\Lambda$ has measure strictly less than 1. Hence, $\partial\Lambda$ has measure zero as required. ☐

3.1 Quasigeodesics from loxodromics

Both S and M are compact, and their fundamental groups are torsion free. The hyperbolic metrics on S and M give maps $\rho_S\colon \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^2)$ and $\rho_M\colon \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$, whose images are torsion-free cocompact lattices. In particular, in both cases, every non-trivial element of the fundamental group maps to a loxodromic isometry. We now show that the image of an axis for a non-trivial element of $\pi_1(S)$ is a quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$.

Proposition 42. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations, and let g be a non-trivial element of $\pi_1(S)$ with axis α in $\widetilde{S}_h$. Then there are constants $Q_\alpha \geqslant 1, c_\alpha$ and K_α such that the image of the axis $\iota(\alpha)$ in $\widetilde{S}_h \times \mathbb{R}$*

is (Q_α, c_α)-quasigeodesic. In particular, the geodesic $\overline{\alpha}$ connecting the endpoints of $\iota(\alpha)$ is contained in a K_α-neighborhood of S_0.

Proof. The isometry $\rho_S(g) \in \mathrm{Isom}(\mathbb{H}^2)$ is loxodromic. Let α be the axis of $\rho_S(g)$ in $\mathbb{H}^2$. By abuse of notation, we will also write g for the image of g in $\pi_1(M)$ under the (injective) inclusion map $i\colon \pi_1(S) \to \pi_1(M)$. Then $\rho_M(g) \in \mathrm{Isom}(\mathbb{H}^3)$ is also loxodromic. As $\widetilde{S}_h \times \mathbb{R}$ is quasi-isometric to $\mathbb{H}^3$, g also acts loxodromically on $\widetilde{S}_h \times \mathbb{R}$. Recall that $\iota\colon \widetilde{S}_h \to \widetilde{S}_h \times \mathbb{R}$ is the inclusion map. The image $\iota(\alpha)$ is an $\rho_M(g)$-invariant path in $\widetilde{S}_h \times \mathbb{R}$. Hence, for constants (Q_α, c_α) that depend on α, the path $\iota(\alpha)$ is a (Q_α, c_α)-quasigeodesic. We shall write $\overline{\alpha}$ for the geodesic in $\widetilde{S}_h \times \mathbb{R}$ connecting the endpoints of $\iota(\alpha)$. It follows that $\overline{\alpha}$ is the axis for g acting on $\widetilde{S}_h \times \mathbb{R}$. By the Morse lemma, Lemma 23, there is a constant $K_\alpha > 0$ such that $\iota(\alpha)$ is contained in an K_α-neighborhood of $\overline{\alpha}$. As $\iota(\alpha)$ is contained in S_0, the proposition follows. □

The axis α projects to a closed geodesic in S_0. Let L_α be the length of this geodesic. Thus, L_α is the translation length of g on $\widetilde{S}_h$. We will use the fact that there is an upper bound P_α on the length of the nearest point projection interval $p_\alpha(\ell)$, for any leaf ℓ of an invariant lamination. Note that we may replace P_α by a larger constant to assume that $P_\alpha \geqslant L_\alpha$, and it will be convenient to do so.

Proposition 43. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Let g be a non-trivial element of $\pi_1(S)$ with axis α in $\widetilde{S}_h$. Then there is a constant $P_\alpha > 0$ such that for any leaf ℓ of either of the invariant laminations, the nearest point projection of ℓ to α and α to ℓ has length at most P_α.*

Proof. Every closed geodesic in S_0 intersects both invariant laminations. Hence, any segment of α with length L_α intersects both laminations Λ_+ and Λ_-. By compactness, there is also a minimum angle $\theta_\alpha > 0$ for an intersection of α with any leaf of an invariant lamination. The above two properties imply that there is an upper bound P_α for the length of the nearest point projection interval $p_\alpha(\ell)$, for any leaf ℓ of an invariant lamination, and similarly for $p_\ell(\alpha)$. □

We now show that the limit points of $\iota(\alpha)$ are disjoint from the limit sets of any ladder $F(\ell)$ of any leaf ℓ of an invariant lamination.

Proposition 44. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that g is a non-trivial element of $\pi_1(S)$. For any leaf ℓ of an invariant lamination, the fixed points $\overline{\alpha}_+$ and $\overline{\alpha}_-$ of $\rho_M(g)$ are disjoint from the limit points of $F(\ell)$ in $\partial(\widetilde{S}_h \times \mathbb{R})$.*

Proof. For an ideal complementary region C of an invariant lamination, we denote by $s(C)$ the number of sides of C. Since there are finitely many ideal complementary regions, the numbers $s(C)$ over all C has a maximum which we denote by $s_{\max}$.

Let ℓ be a leaf of an invariant lamination. Breaking symmetry, suppose that ℓ lies in Λ_+. By Proposition 43, the projection interval $p_\alpha(\ell)$ has length at most P_α. As $\rho_S(g)$ acts on α by translation, we may choose n large enough such that the projection interval $p_\alpha(\rho_S(g)^n \ell)$ is distance greater than $s_{\max} P_\alpha$ from the projection interval $p_\alpha(\ell)$.

Suppose that ℓ and $\rho_S(g)^n \ell$ intersect a common ideal complementary region of the other lamination Λ_-. Traversing in a cyclic order, we may choose a cyclic sequence of boundary geodesics $\ell'_1, \cdots, \ell'_j$ such that ℓ'_1 intersects ℓ and ℓ'_j intersects $\rho_S(g)^n \ell$. But then the projection intervals $p_\alpha(\ell)$ and $p_\alpha(\rho_S(g)^n \ell)$ are at most $jK_\alpha < s_{\max} K_\alpha$ distance apart, a contradiction. We deduce that ℓ and $\rho_S(g)^n \ell$ do not intersect a common ideal complementary region of Λ_-.

On the other hand, since $P_\alpha > L_\alpha$, there is a leaf of Λ_+ separating ℓ and $\rho_S(g)^n \ell$.

By Proposition 19 and Proposition 20, the distance between $F(\ell)$ and $F(\rho_S(g)^n \ell)$ is at least $\epsilon > 0$. By Proposition 21, their limit sets are disjoint. The ladder $F(\rho_S(g)^n \ell)$ equals $\rho_M(g)^n F(\ell)$ and thus the limit set of $F(\ell)$ cannot contain any fixed points of $\rho_M(g)$, as required. □

In a similar vein, we now show that there is an upper bound $P_{\overline{\alpha}}$ on the nearest point projection in $\widetilde{S}_h \times \mathbb{R}$ of the axis $\overline{\alpha}$ to any ladder $F(\ell)$ over any leaf ℓ of an invariant lamination.

Corollary 45. *Suppose that $f: S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that g is a non-trivial element of $\pi_1(S)$ with axis $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. Then there is a constant $P_{\overline{\alpha}} > 0$ such that for any leaf ℓ of an invariant lamination, the diameter of the projection image $p_{\overline{\alpha}}(F(\ell))$ is at most $P_{\overline{\alpha}}$.*

Proof. Suppose that there is a sequence of leaves ℓ_n of the invariant laminations such that the diameters of the projections $p_{\overline{\alpha}}(F(\ell_n))$ tend to infinity as $n \to \infty$. By Proposition 42, the image $\iota(\alpha)$ is contained in a bounded neighborhood of $\overline{\alpha}$. Hence, the diameters of the images of the ladders $F(\ell_n)$ under the nearest point projection to $\iota(\alpha)$, also tend to infinity. As the ladders $F(\ell_n)$ are quasiconvex, the diameters of the subsets of $\iota(\alpha)$ contained in a bounded neighborhood of $F(\ell_n)$ tends to infinity. Hence, the diameters of the nearest point projections of $\ell_n \times \{0\}$ to $\iota(\alpha)$ tends to infinity, contradicting Proposition 43 which states that the diameters are bounded above by P_α. □

We now show that if two points x and y in $\widetilde{S}_h$ have nearest point projections to α which are far apart, then their images $\iota(x)$ and $\iota(y)$ in $\widetilde{S}_h \times \mathbb{R}$ have nearest point projections to $\overline{\alpha}$ which are far apart. As the inclusion map distorts distances, *a priori*, the inclusion of the nearest point projection of x to α need not be close to the nearest point projection of $\iota(x)$ to $\overline{\alpha}$.

Let p_α be the nearest point projection map to α in $\widetilde{S}_h$, and let $p_{\overline{\alpha}}$ be the nearest point projection map to $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. We will use the fact that nearest point projection is also defined for points in the boundary.

Proposition 46. *Suppose that $f: S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that g is a non-trivial element of $\pi_1(S)$, with axis α in $\widetilde{S}_h$, and axis $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. Then there are constants $Q \geqslant 1$ and $c \geqslant 0$ such that for any two points x and y in $\widetilde{S}_h \cup \partial \widetilde{S}_h$,*

$$d_{\widetilde{S}_h \times \mathbb{R}}(p_{\overline{\alpha}}(\iota(x)), p_{\overline{\alpha}}(\iota(y))) \geqslant \frac{1}{Q} d_{\widetilde{S}_h}(p_\alpha(x), p_\alpha(y)) - c.$$

Proof. Let $I = [p, q]$ be the subinterval of α with endpoints $p := p_\alpha(x)$ and $q := p_\alpha(y)$. We may assume that $d_{\widetilde{S}_h}(p, q) \geqslant 2L_1$, where $L_1 = L_\alpha + 2P_\alpha$, where L_α is the translation length of g, and P_α is the largest size of the projection of any leaf of an invariant lamination to α, from Proposition 43. We shall choose $Q = Q_\alpha$ and $c = 2L_1/Q_\alpha + c_\alpha + 2K_\alpha + 2P_{\overline{\alpha}}$, where Q_α and c_α are the quasigeodesic constants for $\iota(\alpha)$ in $\widetilde{S}_h \times \mathbb{R}$ from Proposition 42, and K_α is the corresponding Morse constant.

Let β be a segment of I of length L_1. The central segment of β of length L_α intersects leaves of both laminations. Suppose that ℓ is such a leaf. By our choice of L_1, the projection interval $p_\alpha(\ell)$ is contained in the interior of β, which in turn is contained in I. Let γ be the geodesic spanned by x and y. By Proposition 27, as $p_\alpha(\ell) \subset p_\alpha(\gamma)$, and as α and ℓ intersect, ℓ and γ also intersect.

Let β_1 be an initial segment of I of length L_1, and let β_2 be a terminal segment of I of length L_1. Since $I|I| \geqslant 2L_1$, the segments β_1 and β_2 are disjoint. By the conclusions of the previous paragraph, there exists leaves ℓ_1 and ℓ_2 of an invariant lamination such that

- ℓ_1 intersects β_1 and γ and $p_\alpha(\ell_1)$ lies in the interior of β_1, and
- ℓ_2 intersects β_2 and γ and $p_\alpha(\ell_2)$ lies in the interior of β_2.

It follows that ℓ_1 and ℓ_2 are disjoint and divide $\widetilde{S}_h$ into three regions A_1, A_2 and A_3 such that

- A_1 is adjacent to ℓ_1 and contains x,
- A_2 lies between ℓ_1 and ℓ_2, and

- A_3 is adjacent to ℓ_2 and contains y.

The leaves ℓ_1 and ℓ_2 divide α into arcs α_1, α_2, and α_3 such that α_i is contained in A_i. Furthermore, $p_\alpha(x)$ is contained in α_1, and $p_\alpha(y)$ is contained in α_3. The arc $I \setminus (\beta_1 \cup \beta_2)$ is contained in α_2 and its length is $d_{\widetilde{S}_h}(p,q) - 2L_1$. It follows that the length of α_2 is at least $d_{\widetilde{S}_h}(p,q) - 2L_1$.

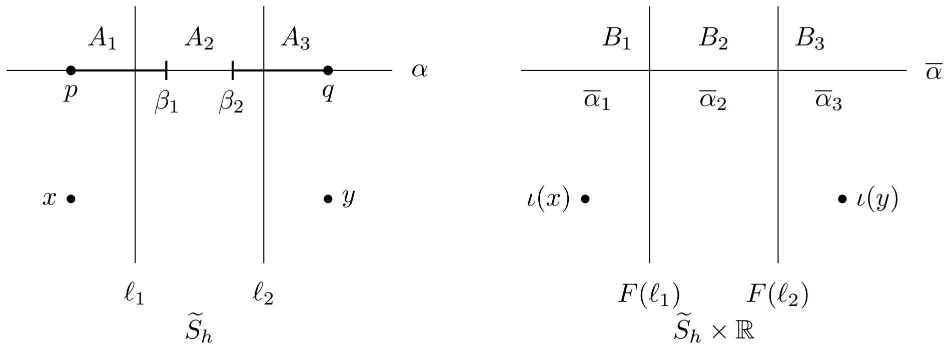

Figure 3: Notation for the complements of the leaves ℓ_i and ladders $F(\ell_i)$.

The ladders $F(\ell_1)$ and $F(\ell_2)$ are convex in $\widetilde{S}_h \times \mathbb{R}$. Since ℓ_1 and ℓ_2 are disjoint, $F(\ell_1)$ and $F(\ell_2)$ are also disjoint. Thus they similarly divide $\widetilde{S}_h \times \mathbb{R}$ into three regions B_1, B_2 and B_3 where

- B_1 is adjacent to $F(\ell_1)$,
- B_2 is between $F(\ell_1)$ and $F(\ell_2)$, and
- B_3 is adjacent to $F(\ell_2)$.

Note that $\iota(A_i) \subseteq B_i$. This implies that the initial limit point of $\overline{\alpha}$ is contained in the limit set of B_1, and the terminal limit point of $\overline{\alpha}$ is contained in the limit set of B_3. By Proposition 44, the limit points of $\overline{\alpha}$ are not contained in the limit points of either $F(\ell_1)$ or $F(\ell_2)$, so the ladders $F(\ell_i)$ also divide $\overline{\alpha}$ into three components, $\overline{\alpha}_1$, $\overline{\alpha}_2$ and $\overline{\alpha}_3$ so that $\overline{\alpha}_i \subset B_i$. Since $\iota(x)$ lies in B_1 its nearest point projection to $\overline{\alpha}$ is contained in the nearest point projection of B_1 to $\overline{\alpha}$. Since the boundary of B_1 is the ladder $F(\ell_1)$, which is convex, we deduce that

$$p_{\overline{\alpha}}(\iota(x)) \subseteq \overline{\alpha}_1 \cup p_{\overline{\alpha}}(F(\ell_1)).$$

In particular, the projection of $\iota(x)$ to $\overline{\alpha}$ is contained in a $P_{\overline{\alpha}}$-neighborhood of α_1. Similarly, the projection of $\iota(y)$ to $\overline{\alpha}$ is contained in $\overline{\alpha}_3 \cup p_{\overline{\alpha}}(F(\ell_2))$, and so is contained in a $P_{\overline{\alpha}}$-neighborhood of $\overline{\alpha}_3$.

The intersection points of ℓ_1 and ℓ_2 with α are endpoints of α_2, whose length is at least $|I| - 2L_1$. Since $\iota(\alpha)$ is a (Q_α, c_α)-quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$, the distance between the intersection points of $F(\ell_1)$ and $F(\ell_2)$ with $\iota(\alpha)$ is at least

$$\frac{1}{Q_\alpha}(|I| - 2L_1) - c_\alpha.$$

Being a quasigeodesic, $\iota(\alpha)$ is contained in an K_α-neighborhood of $\overline{\alpha}$. Therefore, the distance between the nearest point projections of $F(\ell_1)$ and $F(\ell_2)$ to $\overline{\alpha}$ is at least

$$\frac{1}{Q_\alpha}(\mathrm{diam}(|I| - 2L_1) - c_\alpha - 2K_\alpha.$$

By Corollary 45, the diameter of $p_{\overline{\alpha}}(F(\ell_1))$, and similarly $p_{\overline{\alpha}}(F(\ell_2))$, is at most $P_{\overline{\alpha}}$. It follows that the distance in $\widetilde{S_h} \times \mathbb{R}$ between $p_{\overline{\alpha}}(F(\ell_1))$ and $p_{\overline{\alpha}}(F(\ell_2))$ is at least

$$\frac{1}{Q_\alpha}(|I| - 2L_1) - c_\alpha - 2K_\alpha - 2P_{\overline{\alpha}} = \frac{1}{Q} d_{\widetilde{S}_h}(p, q) - c,$$

where $Q = Q_\alpha$ and $c = 2L_1/Q_\alpha + c_\alpha + 2K_\alpha + 2P_{\overline{\alpha}}$. Therefore, the distance between the projections of $\iota(x)$ and $\iota(y)$ to $\overline{\alpha}$ is at least $\frac{1}{Q}\text{diam}(p_\alpha(\gamma)) - c$, where the constants Q and c depend only on α and not on x and y, as required. □

We will also use the well known fact that for a discrete group of isometries of $\mathbb{H}^n$, for any loxodromic element g with axis α, the size of the projections of all of the translates of the axis α to α is bounded, see for example [BBF15, Example 2.1(1)].

Proposition 47. *Suppose that G is a countable group acting locally finitely by loxodromics on $\mathbb{H}^n$. Suppose that g is a loxodromic element with geodesic axis α. Then there is a constant L such that for any distinct translate $h\alpha \neq \alpha$ of the axis, the size of the projection interval $p_\alpha(h\alpha)$ is at most L.*

3.2 Lebesgue measure

In this section we will consider geodesics chosen using Lebesgue measure on either $\partial\widetilde{S}_h$ or $\partial(\widetilde{S}_h \times \mathbb{R})$. We start with the surface case, and show that for the pushforward of Lebesgue measure on $\partial\widetilde{S}_h$, almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$ spend a positive proportion of time close to the base fiber S_0. This shows the Lebesgue measure case of Theorem 6.

Lemma 48. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that ν is the pushforward of Lebesgue measure on $\partial\widetilde{S}_h$ under the Cannon-Thurston map. Then there are constants $R \geqslant 0$ and $\epsilon > 0$ such that for $(\nu \times \nu)$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$,*

$$\lim_{T\to\infty} \frac{1}{T} |\{t \in [0, T] \mid \overline{\gamma}(t) \in N_R(S_0)\}| \geqslant \epsilon.$$

We give a brief overview of the argument. Let α be a geodesic in $\widetilde{S}_h$ which covers a closed geodesic β in S_h. By the results of the previous section, if a geodesic γ in $\widetilde{S}_h$ has a large fellow travel with α, then $\overline{\gamma}$ has a large fellow travel with $\overline{\alpha}$. By ergodicity, a typical geodesic γ in S_h spends a positive proportion of time fellow-traveling any closed geodesic β in S_h. This implies that $\overline{\gamma}$ spends a positive proportion of time close to S_0. We now give a precise version of this argument.

Proof. Let α be a geodesic in $\widetilde{S}_h$ which projects to a closed geodesic β in S_h. By abuse of notation, we denote the lift of β to $T^1(S_h)$ also by β. Let δ_1 be a sufficiently small constant such that $N_{\delta_1}(\beta)$ is a regular neighborhood of β in $T^1(S_h)$. In particular, this implies that δ_1 is less than the injectivity radius of $T^1(S_h)$. We fix a constant $L > 0$ such that

$$L \geqslant Qc + 11\delta_3 + 1, \tag{2}$$

where Q and c are constants in Proposition 46, and δ_3 is the constant of hyperbolicity for $\widetilde{S}_h \times \mathbb{R}$.

Let η be an oriented geodesic in $\widetilde{S}_h$ which intersects $N_{\delta_1}(\alpha)$ in a segment of length L, and let v be the tangent vector to η at the first point of intersection between η and $N_{\delta_1}(\alpha)$. By abuse of notation, we shall also write v for its image in the quotient $T^1(S_h)$.

Let $A(r)$ be the disc of radius r centered at v, perpendicular to the lift of η in $T^1(S_h)$. For any constant $\epsilon > 0$ there is a constant δ such that the forward image of any point w in $A(r)$ under the geodesic flow intersects $N_{\delta_1}(\beta)$ in a segment of length L, up to additive error at most ϵ, and furthermore w intersects $N_{\delta_1}(\beta)$

within distance ϵ under the geodesic flow. Let δ_2 be the value of δ corresponding to $\epsilon_2 = \min\{\delta_1, \delta_3, L/2\}$. We can replace δ_2 by any smaller positive number, so in particular we may assume $\delta_2 \leqslant \delta_3$.

Now choose V to be the image of $A(\delta_2)$ under the forward and backward geodesic flows of distance $\frac{1}{2}\delta_2$. The interior of V is an open regular neighborhood of v. Every geodesic flow line that intersects V intersects it in a segment of length δ_2, and intersects $N_\delta(\beta)$ in a segment of length at least $L - \epsilon_2$, and at most $L + \epsilon_2$, starting within distance ϵ_2 of the intersection of the flow line with V. As $N_{\delta_2}(\beta)$ is a regular neighborhood of β, for any flow line γ, segments of $\gamma \cap N_{\delta_2}(\beta)$ corresponding to distinct intersections with V do not intersect.

As the geodesic flow is ergodic, almost all geodesic flow lines spend a positive proportion of time in V, i.e.

$$\lim_{T\to\infty} \frac{1}{T}|\gamma(0,T) \cap V| = \mathbf{vol}_{T^1(S_h)}(V).$$

As geodesic flow lines intersect V in segments of length δ_2, the number of times a geodesic flow line intersects V is then

$$\lim_{T\to\infty} \frac{1}{T}(\#\text{segments of } \gamma(0,T) \cap V) = \frac{1}{\delta_2}\mathbf{vol}_{T^1(S_h)}(V).$$

Each segment of $\gamma \cap V$ is within distance at most ϵ_2 of a segment of $\gamma \cap N_{\delta_1}(\beta)$ of length at least $L - \epsilon_2$. Therefore, for any interval $\gamma(I)$ containing the segment, the nearest point projection of $\gamma(I)$ to α has length at least $L - \epsilon_2 - 2\delta_1$. By Proposition 46, the nearest point projection of $\overline{\gamma}(\overline{I})$ to $\overline{\alpha}$ is therefore at least $\epsilon_3 = \frac{1}{Q}(L - \epsilon_2 - 2\delta_1) - c$. As both ϵ_2 and δ_1 are at most δ_3, our choice of L from (2) implies $\epsilon_3 \geqslant 8\delta_3 + 1 > 0$.

By Proposition 24, there is a subset of $\overline{\gamma}$ of diameter $\frac{1}{Q}L - c$ contained in an $6\delta_3$-neighborhood of $\overline{\alpha}$. By Proposition 42, $\overline{\alpha}$ is contained in a K_α-neighborhood of S_0. The nearest point projection map from $\iota(\gamma)$ to $\overline{\gamma}$ is distance decreasing, so $\overline{\gamma}$ spends a positive proportion of its length within distance $K_\alpha + 6\delta_3$ of S_0, as required. □

We show that for almost all geodesics chosen according to Lebesgue measure on $\partial(\widetilde{S}_h \times \mathbb{R})$, the proportion of their length which lies close to S_0 tends to zero. This gives the Lebesgue measure case of Theorem 8.

Lemma 49. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that ν is the Lebesgue measure on $\partial(\widetilde{S}_h \times \mathbb{R})$. Then for any constant $R > 0$, for $(\nu \times \nu)$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$,*

$$\lim_{T\to\infty} \frac{1}{T}|\,\{t \in [0,T] \mid \overline{\gamma}(t) \in N_R(S_0)\}\,| = 0.$$

We deduce Lemma 49 directly from the work of Oh and Pan [OP19], which we now describe. We state a special case of their main result which will suffice here. The fibering $M \to S^1$ induces a homomorphism $\pi_1(M) \to \mathbb{Z}$, whose kernel is isomorphic to $\pi_1(S)$. Corresponding to the kernel, we get the $\mathbb{Z}$-cover $M_{\mathbb{Z}}$ homeomorphic to $S \times \mathbb{R}$. The hyperbolic metric on M lifts to a $\mathbb{Z}$-periodic hyperbolic metric on $M_{\mathbb{Z}}$.

The hyperbolic metric on M gives a cocompact lattice Γ_0 in $G = \mathrm{PSL}(2,\mathbb{C})$, so that $\Gamma_0\backslash G$ is the frame bundle for M. The frame flow is denoted by right multiplication by $a_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$, which projects to the geodesic flow in $\mathbb{H}^3$. Haar measure is a left-invariant measure on G, and determines a measure on $\Gamma_0\backslash G$. The frame flow is ergodic with respect to Haar measure, which projects to the Liouville measure on $T^1(M)$. Oh and Pan show the following mixing result for the geodesic flow.

Theorem 50. [OP19, Theorem 1.7.] *Let Γ_0 be a cocompact lattice in $G = PSL(2,\mathbb{C})$, and let $\Gamma\backslash G$ be a $\mathbb{Z}$-cover of $\Gamma_0\backslash G$. Let ψ_1 and ψ_2 be continuous functions on $\Gamma\backslash G$ with compact support. Then*

$$\lim_{t\to+\infty} t^{1/2} \int_{\Gamma\backslash G} \psi_1(xa_t)\psi_2(x)\; dx = \frac{1}{(2\pi\sigma)^{1/2}} \int_{\Gamma\backslash G} \psi_1 \; dx \int_{\Gamma\backslash G} \psi_2 \; dx, \tag{3}$$

where σ is a constant depending on Γ_0.

In particular, the above result implies that the proportion of geodesics starting at height zero (in $M_{\mathbb{Z}}$) that are close to height zero at time t decays at rate $1/\sqrt{t}$. This implies that a typical Lebesgue geodesic in $M_{\mathbb{Z}}$ is recurrent on the fibers, but the proportion of time spent near a fixed fiber decays like $1/\sqrt{t}$.

Proof of Lemma 49. Any choice of homeomorphism g from the mapping torus M to the hyperbolic manifold $\mathbb{H}^3/\Gamma$ lifts to a quasi-isometry $\widetilde{g}$ from $\widetilde{S}_h \times \mathbb{R}$ with the Cannon-Thurston metric to $\mathbb{H}^3$ with the standard metric. We shall write $d_{\mathbb{H}^3}$ for the pullback of the hyperbolic metric to $\widetilde{S}_h \times \mathbb{R}$ by $\widetilde{g}$.

Suppose that p is a basepoint at height zero in $\widetilde{S}_h \times \mathbb{R}$. Let ϕ be a rotationally symmetric continuous approximation to the indicator function of p, normalized so that it integrates to one, with respect to the hyperbolic metric $d_{\mathbb{H}^3}$. Let ϕ' be the pull back of ϕ to $G = \mathrm{PSL}(2, \mathbb{C})$. Let ϕ'_t be the composition of ϕ' with the frame flow, that is $\phi'_t(x) = \phi'(xa_t)$. Then the forward and backward projections of ϕ'_t to $\widetilde{S}_h \times \mathbb{R}$ converge to Lebesgue measure on the sphere at infinity $\partial(\widetilde{S}_h \times \mathbb{R})$.

In applying Theorem 50, we set ψ_1 to be the push forward of ϕ' to $\Gamma\backslash G$. We set ψ_2 to be a close approximation of the indicator function for the pre-image of height-zero fiber in $\Gamma\backslash G$. With this choice, the left hand side integral in (3) gives the proportion of geodesics which at time t lie close to the base fiber in $M_{\mathbb{Z}}$. By (3), this proportion goes to zero at rate $1/\sqrt{t}$. As the hyperbolic metric $d_{\mathbb{H}^3}$ is quasi-isometric to the Cannon-Thurston metric $d_{\widetilde{S}_h \times \mathbb{R}}$, this also holds for the Cannon-Thurston metric. □

3.3 Hitting measure

In this section we consider geodesics chosen according to hitting measure determined by random walks.

We start by showing that for the pushforward of a hitting measure on $\partial\widetilde{S}_h$ arising from a random walk, almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$ spend a positive proportion of time close to the base fiber S_0. This completes the proof of Theorem 6 by showing the hitting measure case for surface measures.

Lemma 51. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that ν and $\check{\nu}$ are the forward and backward hitting measures on $\partial\widetilde{S}_h$ arising from a nonelementary, full random walk on $\pi_1(S)$. Let $\iota_*\nu$ and $\iota_*\check{\nu}$ be their pushforwards under the Cannon-Thurston map. Then there are constants $R \geqslant 0$ and $\epsilon > 0$ such that for $(\iota_*\nu \times \iota_*\check{\nu})$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T} |\{t \in [0, T] \mid \overline{\gamma}(t) \in N_R(S_0)\}| \geqslant \epsilon.$$

Proof. Let γ be a bi-infinite geodesic arising from the limit points of a bi-infinite random walk. We fix a constant $L > 0$ such that $\frac{L}{Q} - c > 8\delta_3$, where Q and c are constants in Proposition 46. Suppose that g is a non-trivial element of $\pi_1(S)$, with axis α in $\widetilde{S}_h$, and axis $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. As open sets have positive hitting measure, there is a positive probability that the length of the projection interval $p_\alpha(\gamma)$ in $\widetilde{S}_h$ has length at least L. By Proposition 46, the diameter of the projection image $p_{\overline{\alpha}}(\overline{\gamma})$ in $\widetilde{S}_h \times \mathbb{R}$ has diameter at least $\frac{1}{Q_\alpha}L - c$. As $\frac{1}{Q_\alpha}L_1 - c > 8\delta_3$, Proposition 24 implies the projection image $p_{\overline{\alpha}}(\overline{\gamma})$ is contained in a $6\delta_3$-neighborhood of $\overline{\gamma}$. So there are points on $\overline{\gamma}$ distance $\frac{1}{Q_\alpha}L_1 - c - 8\delta_3$ apart, such that the interval between them is contained in a $6\delta_3$-neighborhood of $\overline{\alpha}$. By ergodicity, this happens linearly often for translates of α. As the projection map $p_{\overline{\gamma}}$ from $\iota(\gamma)$ to $\overline{\gamma}$ is distance decreasing, $\overline{\gamma}$ spends a positive proportion of time within distance $6\delta_3$ of S_0. □

We now show that for hitting measure on $\partial(\widetilde{S}_h \times \mathbb{R})$ arising from a geometric random walk on $\pi_1(M)$, for almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$, the proportion of their length which is close to the base fiber S_0 tends to zero. This completes the proof of Theorem 8 by showing the hitting measure case for 3-manifold measures. Theorems 6 and 8 then imply Theorem 4.

Proposition 52. *Suppose that $f: S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose ν and $\check{\nu}$ are the forward and backward hitting measures on $\partial(\widetilde{S}_h \times \mathbb{R})$ arising from a geometric random walk on $\pi_1(M)$. Then for $(\nu \times \check{\nu})$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$, and any constant $R > 0$,*

$$\lim_{T\to\infty} \frac{1}{T} |\{t \in [0, T] \mid \overline{\gamma}(t) \in N_R(S_0)\}| = 0.$$

We shall use the following estimate, which is a consequence of the Local Central Limit Theorem for random walks on $\mathbb{Z}$, see for example [LL10, Section 2].

Proposition 53. [LL10, Proposition 2.4.4] *Suppose $\phi_*\mu$ generates an aperiodic random walk on $\mathbb{Z}$. Then there is a constant C such that for all n and x,*

$$\mathbb{P}(\phi(w_n) = x) \leqslant \frac{C}{\sqrt{n}}.$$

Proof of Proposition 52. We sketch the key steps before giving the technical details.

- By the epimorphism $\phi : \pi_1(M) \to \mathbb{Z}$, the μ-random walk projects to an aperiodic random walk on $\mathbb{Z}$.
- By Proposition 53, the random walk on $\mathbb{Z}$ recurs to $O(\log n)$ neighborhoods of zero with negligible probability (as $n \to \infty$).
- The epimorphism ϕ is distance non-increasing for the Cannon-Thurston metric. So we deduce the same statement for recurrence of μ-random walk to $O(\log n)$ neighborhoods of S_0.
- At the same time, the μ-random walk recurs to a $O(\log n)$ neighborhoods of the tracked geodesic with probability negligibly smaller than 1.
- Choosing the size of the neighborhood of the tracked geodesic a definite proportion smaller than the neighborhood of S_0, we derive the fact that the closest points on the tracked geodesic stay $O(\log n)$ distance away from S_0 with probability negligibly smaller than 1.
- The probability estimates imply a sub-linear upper bound for the expectation of the number of times the tracked geodesic recurs to a $O(\log n)$ neighborhood of S_0.
- Using the expectation bound and linear progress with exponential decay, we conclude the proof of Proposition 52.

Let $\phi_*\mu$ denote the pushforward of μ by the homomorphism $\phi: \pi_1(M) \to \mathbb{Z}$. The μ-random walk projects to a $\phi_*\mu$ random walk on $\mathbb{Z}$. As the support of μ generates $\pi_1(M)$ as a semigroup, the support of $\phi_*\mu$ generates $\mathbb{Z}$ as a semigroup. In particular, by Proposition 53, there is a constant C_1 such that for all n and any $A > 0$,

$$\mathbb{P}(|\phi(w_n)| \leqslant 2A \log n) \leqslant 2AC_1 \log n / \sqrt{n}.$$

As the above inequality holds for all $A > 0$, we may choose

$$A = \frac{1}{2 \log \frac{1}{c}}, \tag{4}$$

where $c < 1$ is the exponential decay constant from Proposition 32.

As the distance between any pair of adjacent fibers $S \times \{n\}$ and $S \times \{n + 1\}$ is equal to one in the Cannon-Thurston metric, $|\phi(w_n)| \geqslant 2A \log n$ implies that $d_{\widetilde{S}_h \times \mathbb{R}}(w_n x_0, S_0) \geqslant 2A \log n$. In particular, it is very likely that $w_n x_0$ is far from S_0, i.e.

$$\mathbb{P}\left(d_{\widetilde{S}_h \times \mathbb{R}}(w_n x_0, S_0) \geqslant 2A \log n\right) \geqslant 1 - \frac{2AC_1 \log n}{\sqrt{n}}. \tag{5}$$

Recall that $\gamma(t_n)$ is a closest point on γ to $w_n x_0$. The Gromov product estimate, Proposition 32, implies that it is likely that $w_n x_0$ is logarithmically close to γ, i.e.

$$\mathbb{P}\left(d_{\widetilde{S}_h\times\mathbb{R}}(w_n x_0, \gamma(t_n)) \leqslant A\log n\right) \geqslant 1 - Kc^{A\log n}. \tag{6}$$

Combining (5) and (6) above, and using the triangle inequality, shows that it is likely that $\gamma(t_n)$ is far from S_0, i.e.

$$\mathbb{P}\left(d_{\widetilde{S}_h\times\mathbb{R}}(\gamma(t_n), S_0) \geqslant A\log n\right) \geqslant 1 - \frac{2AC_1\log n}{\sqrt{n}} - Kn^{A\log c}.$$

Taking the complementary event in the line above shows that it is unlikely that $\gamma(t_n)$ is close to S_0, and using the fact that our choice of A from (4) makes $A\log\frac{1}{c} = 1/2$, gives

$$\mathbb{P}\left(d_{\widetilde{S}_h\times\mathbb{R}}(\gamma(t_n), S_0) \leqslant A\log n\right) \leqslant \frac{2AC_1\log n + K}{\sqrt{n}}.$$

Let X_n be the number of times $\gamma(t_k)$ is within distance $A\log k$ of S_0 for $1 \leqslant k \leqslant n$. Then an elementary integral comparison bound says that there is a constant C_2 such that

$$\mathbb{E}(X_n) \leqslant 2AC_1\sqrt{n}\log n + 2K\sqrt{n} + C_2.$$

For any random variable, the Markov inequality says $\mathbb{P}(X \geqslant t) \leqslant \mathbb{E}(X)/t$, so choosing $t = \sqrt{n}(\log n)^2$ gives

$$\mathbb{P}(X_n \geqslant \sqrt{n}(\log n)^2) \leqslant \frac{2AC_1}{\log n} + \frac{2K}{(\log n)^2} + \frac{C_2}{\sqrt{n}(\log n)^2}. \tag{7}$$

Let $\beta_n = \gamma([t_{n-1}, t_n])$, and set

$$B_n = \bigcup_{1\leqslant k\leqslant n} \beta_k.$$

Recall that by Corollary 34, t_n makes linear progress with exponential decay. In particular, there is a constant $\ell > 0$ such that

$$\mathbb{P}\left(\gamma([0,\ell n]) \subseteq B_n \subseteq \gamma([0, 2\ell n])\right) \geqslant 1 - Kc^n.$$

By Proposition 35, there is a constant $D > 0$ such that the probability that $|\beta_k| \leqslant D\log k$ for all $k \geqslant \log n$ tends to one as n tends to infinity. Let D_n be the union of all β_k, for $1 \leqslant k \leqslant n$, such that any point on β_k is within distance $(A+D)\log k$ of S_0. By (7), the number of such intervals is at most $\sqrt{n}(\log n)^2$, with probability that tends to one as n tends to infinity. The probability that the union of the first $\log n$ segments $B_{\log n}$ has total length at most $2\ell\log n$ tends to one as n tends to infinity. Therefore,

$$\mathbb{P}\left(\frac{|D_n|}{|B_n|} \leqslant \frac{2\ell\log n + D\sqrt{n}(\log n)^3}{\ell n}\right) \to 1 \text{ as } n \to \infty.$$

In particular, the proportion of points in $\gamma([0, t_n])$ which lie within distance K of S_0 tends to zero as n tends to infinity, as required. □

4 Effective bounds for surface measures

In this section, we prove Theorem 7 using several results that we state and prove in this section, and using our construction of quasigeodesics in $\widetilde{S}_h \times \mathbb{R}$, that we set out subsequently in Section 6.

Suppose that γ is a non-exceptional geodesic with unit speed parametrization. The inclusion map ι embeds γ in $\widetilde{S}_h \times \mathbb{R}$ at height $z = 0$. The image $\iota(\gamma)$ is, in general, not a quasigeodesic. However, we will show that the height for each point of $\iota(\gamma)$ can be changed in a specified way so that the resulting path is a quasigeodesic, i.e. there is a function $h_\gamma(t)$ such that $(\gamma(t), h_\gamma(t))$ is an (unparametrized) quasigeodesic.

In order to define the function h_γ, we need the following mild generalization of a measured lamination. A measured lamination is *maximal* if every complementary region is an ideal triangle. By adding finitely many leaves to divide every ideal complementary region of a measured lamination Λ into ideal triangles we may extend Λ to a maximal lamination. There are thus only finitely many such maximal laminations containing Λ. We will call the union of these minimal laminations the *extended lamination* $\overline{\Lambda}$, which in general is not itself a measured lamination. See Definition 71 and Section 5.2 for more details.

Let $\overline{\Lambda}$ be the union of the two extended laminations obtained from the invariant laminations for f, and let $\overline{\Lambda}^1$ be its lift in $T^1(S_h)$. Let $h\colon T^1(S_h)\setminus\overline{\Lambda}^1 \to \mathbb{R}$ be a continuous function. Then h determines an embedding in $\widetilde{S}_h \times \mathbb{R}$ of any non-exceptional unit-speed geodesic γ by the map $t \mapsto (\gamma(t), h(\gamma^1(t)))$, where $\gamma(t)$ is the oriented geodesic and $\gamma^1(t)$ is the unit tangent vector to γ at $\gamma(t)$. We shall call h the *height function*, and the embedding $\tau_\gamma(t) = (\gamma(t), h(\gamma^1(t)))$ the *test path* for γ determined by h.

We now specify the height function we will use, see Section 6 for background and motivation.

We shall write $\log_k$ for the logarithm function with base k, where $k = k_f > 1$ is the stretch factor of f.

Definition 54. Suppose that $f\colon S \to S$ is a pseudo-Anosov map, let (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations, and $\overline{\Lambda}^1_-$ and $\overline{\Lambda}^1_+$ the lifts in $T^1(S_h)$ of the extended laminations given by the invariant laminations of f, and let $\theta > 0$ be a positive constant. We define the *height function* $h_\theta\colon T^1(S_h) \setminus (\overline{\Lambda}^1_+ \cup \overline{\Lambda}^1_-) \to \mathbb{R}$ to be

$$h_\theta(v) = \log_k \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(v, \overline{\Lambda}^1_+)} - \log \frac{1}{\theta} \right\rfloor_1 - \log_k \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(v, \overline{\Lambda}^1_-)} - \log \frac{1}{\theta} \right\rfloor_1$$

Here $\lfloor x \rfloor_c = \max\{x, c\}$ is the standard floor function. As the two extended laminations are a positive distance apart in $T^1(S_h)$, for sufficiently small θ, at most one of the terms on the right hand side above will be non-zero.

We prove that for a choice of θ sufficiently small, the test path determined by the corresponding height function is a quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$.

Theorem 55. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there are constants $\theta > 0$, $Q \geqslant 1$ and $c \geqslant 0$, such that for any non-exceptional geodesic γ in S_h, with a unit speed parametrization $\gamma(t)$, the test path $\tau_\gamma(t) = (\gamma(t), h_\theta(\gamma^1(t)))$ is an unparametrized (Q, c)-quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$ with the same limit points as $\iota(\gamma)$, where h_θ is the height function from Definition 54.*

We shall now fix a sufficiently small constant θ in Theorem 55 and simplify notation to just write h for h_θ. See Section 6.2.2 for the exact choice of θ that we use. Furthermore, we will write $h_\gamma(t)$ for $h_\theta(\gamma^1(t))$.

We will use one further property of these quasigeodesics. The test path τ_γ lies in the ladder $F(\gamma)$ determined by γ, so vertical projection gives a map $(\gamma(t), 0) \mapsto (\gamma(t), h_\gamma(t))$ from $\iota(\gamma)$ to the test path τ_γ. We will prove that this map is coarsely distance non-increasing.

Proposition 56. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. There are constants $K > 0$ and $c \geqslant 0$ such that for any non-exceptional geodesic γ with unit speed parametrization, and any real numbers s and t,*

$$d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(s), \tau_\gamma(t)) \leqslant K d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(s)), \iota(\gamma(t))) + c,$$

where here the test path has the parametrization inherited from the unit speed parametrization on γ.

4.1 Lebesgue measure

Assuming Theorem 55 and Proposition 56, we now derive Theorem 57 giving effective bounds for the amount of time that a geodesic chosen according to Lebesgue measure on the boundary circle of $\widetilde{S}_h$ spends close to the base fiber S_0. This implies the first half of Theorem 7.

Theorem 57. *Suppose that f is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Let ν be the pushforward of Lebesgue measure on $\partial\widetilde{S}_h$ under the Cannon-Thurston map. Then there are constants $K > 0$ and $\alpha > 0$ such that for $(\nu \times \nu)$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$, for any $R \geqslant 0$,*

$$\lim_{T\to\infty} \frac{1}{T} |\overline{\gamma}([0,T]) \cap N_R(S_0)| \geqslant 1 - Ke^{-\alpha k^R}.$$

Here is an overview of the argument.

1. Let $\overline{\Lambda}^1$ be the union of the lifts of both extended laminations in $T^1(S_h)$. By Definition 94 of the height function h_γ, if $|h_\gamma(t)| \geqslant R$, then the tangent vector at $\gamma(t)$ lies in $N_r(\overline{\Lambda}^1)$, for $r = Ke^{-k^R}$.

2. Work of Birman–Series [BS85] shows that the volume of $N_r(\overline{\Lambda}^1)$ goes to zero as $r \to 0$ at the rate $r^2(\log \frac{1}{r})^{6g-6}$.

3. Suppose that γ is a geodesic chosen according to Lebesgue measure on $\partial\widetilde{S}_h$. The geodesic flow on $T^1(S_h)$ is ergodic with respect to Liouville measure, which is the product of the hyperbolic metric on S_h with Lebesgue measure on the unit tangent circles. Therefore almost all geodesics γ are uniformly distributed in $T^1(S_h)$. In particular, we deduce that
   - we may fix R_0 large enough and hence $r_0 = Ke^{-kR_0}$ small enough such that γ recur to $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, and
   - the proportion of time along γ for which $|h_\gamma(t)|$ is at least R is at most $O(e^{-k^R})$, where the time is being measured in terms of a unit speed parametrization of γ in $\widetilde{S}_h$.

4. By the work of Gadre–Hensel [GH24] the distance in $\widetilde{S}_h \times \mathbb{R}$ from $\iota(\gamma)(0)$ to $\iota(\gamma)(T)$ grows linearly in T. From the linear growth and recurrence to $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, we deduce that the distance $d(t)$ between $\tau_\gamma(0)$ and $\tau_\gamma(t)$ also grows linearly in T. From this, we deduce that, as a proportion of $d(T)$, the time along τ_γ for which $|h_\gamma|$ is at least R is again at most $O(e^{-k^R})$.

5. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ. As the test path τ_γ is quasigeodesic, there is a constant L such that the test path is contained in an L-neighborhood of $\overline{\gamma}$. Therefore, parametrizing $\overline{\gamma}$ with unit speed, the proportion of times on $\overline{\gamma}$ outside $N_R(S_0)$ goes to zero at rate $e^{-k^{R-L}}$, as desired.

Step 1 requires no further elaboration. We now justify Step 2.

Let Λ be a geodesic lamination in the surface S_h, and let $N_r(\Lambda)$ be the set of all points in S distance at most r from Λ. Birman and Series [BS85] give the following bounds for the area of $N_r(\Lambda)$.

Theorem 58. [BS85] *Suppose that S is a surface with a complete finite-area hyperbolic metric. Then there are constants $A > 0$ and $r_0 > 0$, such that for any geodesic lamination Λ on S, and for all $r \leqslant r_0$,*

$$\frac{1}{A} r \leqslant \mathbf{area}(N_r(\Lambda)) \leqslant Ar(\log \tfrac{1}{r})^{6g-6}.$$

The upper bound follows from [BS85, Proposition 4.1] with the degree of the exponent given by [BS85, Remark 7.2]. Birman–Series state only the upper bound. The lower bound is immediate from the observation that any geodesic lamination contains an embedded simple arc, and the area of an r-neighborhood of a simple arc is proportional to r for r sufficiently small. The theorem above gives the following immediate bounds on the volumes of an r-neighborhood of Λ in the unit tangent bundle. Let Λ^1 be the pre-image of Λ in the unit tangent bundle $T^1(S)$, and let $N_r(\Lambda^1)$ be the set of all points in $T^1(S)$ distance at most r from Λ^1.

Corollary 59. *Suppose that S is a surface with a complete finite-area hyperbolic metric. Then there are constants $A > 0$ and $r_0 > 0$, such that for any extended geodesic lamination $\overline{\Lambda}$ on S, and all $r \leqslant r_0$,*

$$\frac{1}{A} r^2 \leqslant \mathbf{vol}(N_r(\overline{\Lambda}^1)) \leqslant A r^2 (\log \tfrac{1}{r})^{6g-6}.$$

Proof. A geodesic lamination Λ has finitely many complementary regions and can be extended in finitely many ways to a maximal lamination Λ_i by dividing each complementary region into ideal triangles. The extended lamination $\overline{\Lambda}$ is thus contained in the union of finitely many laminations Λ_i. The result follows immediately from Theorem 58, by replacing A by An, where n is the number of geodesic laminations in the collection Λ_i. □

Step 3 follows from Corollary 59 and ergodicity of the geodesic flow on $T^1(S_h)$. Using Definition 94, we may write a version of Theorem 57 for the test path τ_γ, using the parametrization of γ in $\widetilde{S}_h$, instead of a unit speed parametrization of the test path.

Proposition 60. *Suppose f is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there is a constant $K > 0$ such that for Lebesgue-almost all geodesics γ in $\widetilde{S}_h$, for any unit speed parametrization of the geodesic $\gamma(t)$, for any $R \geqslant 0$,*

$$\lim_{T \to \infty} \frac{1}{T} \left| \tau_\gamma([0,T]) \setminus N_R(S_0) \right| \leqslant K e^{-k^R},$$

where $\tau_\gamma(t)$ is the parametrization induced by $\gamma(t)$, not the arc length parametrization.

Proof. By ergodicity of the geodesic flow, the amount of time almost every geodesic spends within distance r of the union of the extended laminations $\overline{\Lambda}^1$ is equal to the volume of $N_r(\overline{\Lambda}^1)$.

$$\lim_{T \to \infty} \frac{1}{T} \left| \gamma^1([0,T]) \cap N_r(\overline{\Lambda}^1) \right| = \mathbf{vol}(N_r(\overline{\Lambda}^1))$$

By Corollary 59,

$$\lim_{T \to \infty} \frac{1}{T} \left| \gamma^1([0,T]) \cap N_r(\overline{\Lambda}^1) \right| \leqslant A r^2 (\log \tfrac{1}{r})^{6g-6}. \tag{8}$$

By the definition of the test path, Definition 94, if $\gamma(t)$ lies in $N_r(\overline{\Lambda}^1)$ then the corresponding point on the test path $\tau_\gamma(t)$ lies outside $N_R(S_0)$, where $r = K e^{-k^R}$. Rewriting the upper bound in (8) in terms of R gives

$$\lim_{T \to \infty} \frac{1}{T} \left| \tau_\gamma([0,T]) \setminus N_R(S_0) \right| \leqslant A K^2 e^{-2k^R} k^{(6g-6)R}.$$

Furthermore, for sufficiently large R, $e^{k^R} \geqslant k^{(6g-6)R}$, so for an appropriate choice of $K_1 = AK^2$,

$$\lim_{T \to \infty} \frac{1}{T} \left| \tau_\gamma([0,T]) \setminus N_R(S_0) \right| \leqslant K_1 e^{-k^R}, \tag{9}$$

as required. □

To deduce Step 4, we first state the following result of Gadre–Hensel [GH24] showing that the distance in $\widetilde{S}_h \times \mathbb{R}$ along $\iota(\gamma)$ grows linearly.

Theorem 61. [GH24, Theorem 2.2] *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there is a constant $\epsilon > 0$ such that for Lebesgue-almost all geodesics γ in $\widetilde{S}_h$, with unit speed parametrization, we have*

$$\frac{1}{T} d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(0)), \iota(\gamma(T))) > \epsilon$$

for all T large enough depending on γ.

We deduce from Theorem 61, the recurrence of γ to $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, and the quasi-geodesicity of τ_γ, that the distance in $\widetilde{S}_h \times \mathbb{R}$ along the test path τ_γ also grows linearly with respect to the parametrization coming from γ.

Corollary 62. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there is a constant $\epsilon > 0$ such that for Lebesgue-almost all geodesics γ in $\widetilde{S}_h$, with unit speed parametrization, we have*

$$\frac{1}{T} d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(0), \tau_\gamma(T)) > \epsilon.$$

for all T large enough depending on γ.

Proof. Suppose that γ is a non-exceptional geodesic in $\widetilde{S}_h$ sampled with respect to the Lebesgue measure and parametrized by unit speed. Suppose that the test path τ_γ is given the parametrization from γ.

We may fix R_0 large enough and hence $r_0 = Ke^{-kR_0}$ small enough such that for Lebesgue-almost all γ the time that $\gamma([0,T])$ spends in $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$ is strictly greater than $2/3$ for all T large enough. Suppose that $t_0 > 0$ is the smallest time for which $\gamma(t_0)$ lies in $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, and suppose that $t_1 < T$ is the largest time for which $\gamma(t_1)$ lies in $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$. It follows that $t_0 < T/3$ and $t_1 > 2T/3$, and so $t_1 - t_0 \geqslant T/3$. By the triangle inequality,

$$d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_1)) \geqslant d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \iota(\gamma(t_1))) - d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \tau_\gamma(t_0)) - d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_1)), \tau_\gamma(t_1)).$$

By definition of the test path, the final two terms on the right hand side are equal to the absolute value of the height function, and so

$$\begin{aligned} d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_1)) &\geqslant d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \iota(\gamma(t_1))) - |h_\gamma(t_0)| - |h_\gamma(t_1)| \\ &\geqslant d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \iota(\gamma(t_1))) - R_0 - R_0. \end{aligned}$$

By Theorem 61, $d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \iota(\gamma(t_1))) \geqslant \epsilon(t_1 - t_0)$, and so

$$d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_1)) \geqslant \epsilon(t_1 - t_0) - 2R_0 \geqslant \frac{1}{3}\epsilon T - 2R_0.$$

We conclude the proof by noting that the test path τ_γ is a quasi-geodesic in $\widetilde{S}_h \times \mathbb{R}$ and hence the distance $d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_1))$ is a coarse lower bound for $d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(0), \tau_\gamma(T))$. ☐

Finally, we prove Step 5, completing the proof of Theorem 57.

Proof (of Theorem 57). We obtain from γ a parametrization $\mathbb{R} \to \overline{\gamma}$ by letting $\overline{\gamma}_t$ be a point of $\overline{\gamma}$ closest to $\tau_\gamma(t)$. Thus $\overline{\gamma}_0$ and $\overline{\gamma}_T$ are points of $\overline{\gamma}$ closest to $\tau_\gamma(0)$ and $\tau_\gamma(T)$ respectively. Since τ_γ is a quasi-geodesic there is a constant L such that $d_{\widetilde{S}_h \times \mathbb{R}}(\overline{\gamma}_0, \tau_\gamma(0)) < L$ and $d_{\widetilde{S}_h \times \mathbb{R}}(\overline{\gamma}_T, \tau_\gamma(T)) < L$. By triangle inequality,

$$d_{\widetilde{S}_h \times \mathbb{R}}(\overline{\gamma}_0, \overline{\gamma}_T) \geqslant d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(0), \tau_\gamma(T)) - 2L.$$

By Corollary 62, there is then an $\epsilon > 0$ such that

$$d_{\widetilde{S}_h \times \mathbb{R}}(\overline{\gamma}_0, \overline{\gamma}_T) \geqslant \epsilon T \tag{10}$$

On the other hand, the projection $\iota(\gamma)$ to τ_γ along flow lines is distance decreasing. Hence, it follows that

$$d_{\widetilde{S}_h \times \mathbb{R}}(\overline{\gamma}_0, \overline{\gamma}_T) \leqslant T + 2L. \tag{11}$$

By Proposition 60, for Lebesgue almost all geodesics γ, there is a $T_1(\gamma)$, such that for all $T \geqslant T_1(\gamma)$,

$$|\tau_\gamma([0,T]) \setminus N_R(S_0)| \leqslant TKe^{-k^R}.$$

As $\overline{\gamma}$ is contained in an L-neighborhood of the test path τ_γ, we deduce

$$|\overline{\gamma}([0,T]) \setminus N_{R+L}(S_0)| \leqslant TKe^{-k^R}$$

where $\overline{\gamma}$ is parametrized from γ. Finally, suppose that $\overline{\gamma}(D) = \overline{\gamma}_T$ in the arc length parametrization of $\overline{\gamma}$ in which $\overline{\gamma}(0) = \overline{\gamma}_0$. Then, by Equation (10),

$$|\overline{\gamma}([0,T]) \setminus N_{R+L}(S_0)| \leqslant \frac{1}{\epsilon} DKe^{-k^R}$$

from which, by tweaking constants, we may deduce Theorem 57, as required. □

4.2 Hitting measure

Again assuming Theorem 55 and Proposition 56, we now derive Theorem 63 giving effective bounds for the proportion of time that a geodesic chosen by a hitting measure on the boundary circle of $\widetilde{S}_h$ spends close to the base fiber S_0. This completes the second half of Theorem 7.

Theorem 63. *Suppose that f is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let ν and $\check{\nu}$ be the forward and backwards hitting measures on $\partial\widetilde{S}_h$. Then there are constants $K > 0$ and $\alpha > 0$ such that for $(\iota_*\nu \times \iota_*\check{\nu})$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$, for any $R \geqslant 0$,*

$$\lim_{T\to\infty} \frac{1}{T} |\overline{\gamma}([0,T]) \setminus N_R(S_0)\}| \leqslant Ke^{-\alpha k^R}.$$

Here is an overview of the argument.

1. Suppose that γ is a geodesic in $\widetilde{S}_h$ chosen according to the hitting measure $\nu \times \check{\nu}$ on $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Suppose that the tangent vector at $\gamma(t)$ is within distance $r > 0$ of one of the extended laminations Λ. If r is very small then the endpoints of γ are within distance $r + o(r)$ of the limit set of Λ, when $\partial\widetilde{S}_h$ is equipped with the angular metric viewed from $\gamma(t)$. Work of Birman–Series [BS85] shows that the hitting measure of $N_r(\partial\Lambda)$ goes to zero at rate r^α.

2. Let x_0 be a choice of basepoint in $\widetilde{S}_h$, and let $\gamma(t_n)$ be the closest point on γ to $w_n x_0$. The previous argument shows that the probability that the tangent vector at $\gamma(t_n)$ is within distance r of one of the extended laminations goes to zero at rate r^α. Let R be such that $r = e^{-k^R}$. By the definition of the height function, the probability that $|h_\gamma(t_n)| \geqslant R$ goes to zero at rate $e^{-\alpha k^R}$.

3. Since μ is assumed to have finite support, the distance in $\widetilde{S}_h$ between any two successive locations $w_n x_0$ and w_{n+1} of the random walk is bounded. As nearest point projection to geodesics is distance reducing, the distance between any two successive nearest point projections $\gamma(t_n)$ and $\gamma(t_{n+1})$ is also bounded. We use the fact that the height function is Lipschitz to show that that the distance in $\widetilde{S} \times \mathbb{R}$ between the corresponding test path locations $\tau_\gamma(t_n)$ and $\tau_\gamma(t_n + 1)$ is also bounded.

4. We use the linear progress/ drift of the random walk in $\widetilde{S}_h \times \mathbb{R}$ to show that the test path locations $\tau_\gamma(t_n)$ also make linear progress in $\widetilde{S}_h \times \mathbb{R}$. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ. We parametrize $\overline{\gamma}$ by unit speed so that $\overline{\gamma}(0)$ is the point closest to the first test path location $\tau_\gamma(t_0)$. As the test path τ_γ is contained in a bounded neighborhood of the geodesic $\overline{\gamma}$, the number of test path locations close to the segment $\overline{\gamma}([0,T])$ grows linearly in T.

5. As the proportion of test path locations outside of $N_R(S_0)$ goes to zero at rate $e^{-\alpha k^R}$, the proportion of time $\overline{\gamma}([0,T])$ spends outside $N_R(S_0)$ goes to zero at the same rate.

We start by estimating hitting measure for a regular neighborhood of the limit set of the extended laminations. We will use the following result from the proof of [BS85, Theorem II, page 224], with the degree of the polynomial from [BS85, Remark 7.2].

Theorem 64. [BS85] *Suppose that S_h is a closed hyperbolic surface and Λ is a geodesic lamination. Let x_0 be a basepoint in $\widetilde{S}_h$. Then there are constants A, c and $\alpha > 0$ such that for any integer $n > 0$, there are An^{6g-g} squares of side length $ce^{-\alpha n}$ which cover $\partial\Lambda$ in $(\partial S_h \times \partial S_h) \setminus \Delta$.*

We use Theorem 64 to estimate the hitting measure of a regular neighborhood of the limit set of the extended laminations.

Corollary 65. *Suppose that f is a pseudo-Anosov map and S_h a hyperbolic metric. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let $\nu \times \check{\nu}$ be the resulting stationary measure on $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Let $\partial\overline{\Lambda}$ be the limit set in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$ of the union of the extended invariant laminations. Choose a basepoint x_0 in $\widetilde{S}_h$, and give $\widetilde{S}_h$ the visual metric based at x_0, and give $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$ the product metric. Then there are constants $K > 0$ and $\alpha > 0$ such that*

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant K r^\alpha$$

Proof. As noted before, the union of the extended laminations $\overline{\Lambda}$ is contained in the union of a finite number of geodesic laminations $\Lambda_1, \ldots \Lambda_k$. Let A_1, c_1 and α_1 be the constants from Theorem 64. Setting $r = c_1 e^{-\alpha_1 n}$ in Theorem 64, each $N_r(\Lambda_i)$ is contained in the union of at most $A_1(\frac{1}{\alpha_1} \log \frac{c_1}{r})^{6g-6}$ squares of side length r. As an r-neighborhood of a square of side length r is contained in a union of 9 squares of side length r, this shows that $N_r(\overline{\Lambda})$ is contained in the union of at most $9A_1 k(\frac{1}{\alpha} \log \frac{c_1}{r})^{6g-6}$ squares of side length r.

The Gromov product of two points in the boundary is equal to $\log(1/\sin(\theta/2))$, where θ is the angle between them viewed from the basepoint, see for example [Roe03, page 114]. Using the elementary bounds $x/2 \leqslant \sin x \leqslant x$ for $0 \leqslant x \leqslant 1$, we know that an interval of length r in $\partial\widetilde{S}_h$ is contained in a shadow of distance $\log \frac{1}{r}$ from the basepoint. By exponential decay of shadows, Proposition 32, there is a constant $c_2 < 1$ such that the hitting measure of an interval of length r is at most $Kc_2^{\log \frac{1}{r}} = Kr^\beta$ for some $\beta = \log \frac{1}{c_2} > 0$. So the hitting measure of a square of side length r is at most $K^2 r^{2\beta}$. This gives

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant K^2 r^{2\beta} \; 9A_1(\tfrac{1}{\alpha} \log \tfrac{c_1}{r})^{6g-6}.$$

As $(\log \frac{c_1}{r})^{6g-6}$ is a polynomial in $\log \frac{1}{r}$, there is a constant A_2 such that

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant A_2 r^{2\beta} (\log \tfrac{1}{r})^{6g-6}.$$

As $r^\beta(\log \frac{1}{r})^{6g-6}$ tends to zero as r tends to zero, there is a constant A_3 such that

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant A_3 r^\beta,$$

and so the result follows by setting $K = A_3$ and $\alpha = \beta$. □

Let x_0 be a basepoint for $\widetilde{S}_h$. As the distribution μ generating the random walk has finite support, the distance between any two successive locations $w_n x_0$ and $w_{n+1} x_0$ of the random walk is bounded. As closest point projection to a geodesic is distance reducing, this implies that the distance between the corresponding closest points $\gamma(t_n)$ and $\gamma(t_{n+1})$ on γ is also bounded. We now show that the distance between the corresponding test path locations $\tau_\gamma(t_n)$ and $\tau_\gamma(t_{n+1})$ is bounded.

Proposition 66. *Suppose that f is a pseudo-Anosov map and S_h a hyperbolic metric. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let x_0 be a basepoint for $\widetilde{S}_h$. Let γ be the geodesic with the same limit points as $(w_n x_0)_{n\in\mathbb{Z}}$, and let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to γ. Then there is a constant B such that for all n,*

$$d_{\widetilde{S}\times\mathbb{R}}(\tau_\gamma(t_n), \tau_\gamma(t_{n+1})) \leqslant B.$$

Proof. Let x_0 be a basepoint for $\widetilde{S}_h$, and let $(w_n x_0)_{n\in\mathbb{Z}}$ be a bi-infinite sample path of the random walk. Let γ be the geodesic determined by its limit points of the sample path. By a unit speed parametrization of γ, we get a parametrization of the test path $\tau_\gamma(t)$.

As μ has finite support, there is an upper bound B_1 on the distance between $w_n x_0$ and $w_{n+1} x_0$. Let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to the geodesic γ in $\widetilde{S}_h$. As nearest point projection to a geodesic is distance decreasing, the distance between $\gamma(t_n)$ and $\gamma(t_{n+1})$ is at most B_1. As the inclusion map ι is distance decreasing, the distance between $\iota(\gamma(t_n))$ and $\iota(\gamma(t_{n+1}))$ is also at most B_1.

By Proposition 56, there are constants K and c such that the distance between the corresponding test path locations $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ is at most $B = KB_1 + c$, as required. □

We now show that the distance in $\widetilde{S}_h \times \mathbb{R}$ between the test path locations $\tau_\gamma(t_0)$ and $\tau_\gamma(t_n)$ grows linearly in n.

Proposition 67. *Suppose that f is a pseudo-Anosov map and S_h a hyperbolic metric. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let x_0 be a basepoint for $\widetilde{S}_h$. Let γ be the geodesic with the same limit points as $\{w_n x_0\}$, and let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to γ. Then there is a constant $\ell > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{N} d_{\widetilde{S}\times\mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_N)) = \ell > 0.$$

Proof. Let $\overline{x}_0 = \iota(x_0)$. By the triangle inequality applied to the path consisting of the three geodesic segments $[\overline{x}_0, \tau_\gamma(t_0)] \cup [\tau_\gamma(t_0), \tau_\gamma(t_n)] \cup [\tau_\gamma(t_n), w_n \overline{x}_n]$, we get

$$d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_n)) \geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, w_n\overline{x}_0)) - d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, \tau_\gamma(t_0)) - d_{\widetilde{S}_h\times\mathbb{R}}(w_n\overline{x}_0, \tau_\gamma(t_n)).$$

Set $A = d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, \tau_\gamma(t_0))$, which is independent of n. As the inclusion map ι is distance decreasing, the distance in $\widetilde{S}_h \times \mathbb{R}$ from $w_n\overline{x}_0$ to $\tau_\gamma(t_n)$ is at most the distance in $\widetilde{S}_h$ from $w_n x_0$ to $\gamma(t_n)$, plus the distance from $\iota(\gamma(t_n))$ to $\tau_\gamma(t_n)$, which is given by the value of the height function $h_\gamma(t_n)$. This gives

$$d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_n)) \geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, w_n\overline{x}_0)) - A - d_{\widetilde{S}_h}(w_n x_0, \gamma(t_n)) - h_\gamma(t_n).$$

The random walk makes linear progress in $\widetilde{S}_h \times \mathbb{R}$ at rate ℓ_3, with exponential decay. The random variable $d_{\widetilde{S}_h}(x_0, \gamma(t_0))$ has a distribution with exponential tails. By Corollary 65, the random variable $h_\gamma(t_0)$ also has exponential tails. Hence, as $N \to \infty$, $\frac{d_{\widetilde{S}_h}(w_N x_0, \gamma(t_N))}{N} \to 0$ and $\frac{h_\gamma(t_N)}{N} \to 0$. Therefore

$$\lim_{t \to \infty} \frac{1}{N} d_{\widetilde{S}_h \times \mathbb{R}} \left(\overline{\gamma}(\bar{t}_0), \overline{\gamma}(\bar{t}_N) \right) \geqslant \ell_3 > 0,$$

as required. □

We now record the following elementary properties of visual measure in $\widetilde{S}_h$.

Proposition 68. *Suppose that x_0 is a basepoint for $\mathbb{H}^2$, and ℓ a geodesic. Suppose that p is the closest point of ℓ to x_0. For $r \leqslant \frac{1}{2}$, let γ be a geodesic which intersects an r-neighborhood in $T^1(\mathbb{H}^2)$ of the tangent vector to ℓ at p. Then the endpoints of ℓ and γ in $\partial_\infty \mathbb{H}^2$ are distance at most $8r$ apart.*

Proof. Suppose ℓ passes through x_0 so that $p = x_0$. Suppose that γ also passes through x_0 and makes an angle at most r with ℓ. Then the distance between their endpoints in the visual metric on $\partial_\infty \mathbb{H}^2$ from x_0, is at most r.

Now suppose that

- ℓ and γ are disjoint, and
- x_0 is the midpoint of the shortest geodesic segment α between ℓ and γ.

Let β be the perpendicular bisector to α at x_0. Choose an endpoint in $\partial_\infty \mathbb{H}^2$ of γ and let q be the nearest point projection of that endpoint to β. In fact, the geodesic between the endpoint and q extends to a bi-infinite geodesic perpendicular to β at q. Consider the right angled triangle with vertices x_0, q and the chosen endpoint of γ as an ideal vertex. Let t be the distance from x_0 to q. By Proposition 25, $t \geqslant \log \frac{2}{r}$. The angle at x_0 is half the angle between the endpoints of ℓ and γ at x_0. By the tangent formula for right angled hyperbolic triangles, $\tan(\theta/2) = 1/\sinh(t)$. Using the elementary bounds that $x \leqslant \tan(x) \leqslant 2x$ for $0 \leqslant x \leqslant 1$, and the bound on t in terms of r, we get $x \leqslant 8r/(4 - r^2)$. As we have assumed that $r \leqslant \frac{1}{2}$, this implies that $\theta/2 \leqslant 4r$, as required.

Finally, suppose that in the case that the geodesics intersect, x_0 is not the point of intersection, and if the geodesics do not intersect, x_0 is not the midpoint of the shortest geodesic segment between them. In the former case, we may fix some isometry g that moves x_0 to the intersection point. Restricted to small intervals about the endpoints of $g^{-1}\gamma$ and their images by g, the isometry g is a contraction for the visual metric from x_0. In the case that γ and ℓ do not intersect, we consider an isometry that moves x_0 to the midpoint of the shortest geodesic segment between γ and ℓ. Again, the isometry restricts to a contraction near the endpoints of $g^{-1}(\gamma)$, and so the result follows. □

We now prove Theorem 63.

Proof (of Theorem 63). We fix a basepoint x_0 in $\widetilde{S}_h$ and give $\partial \widetilde{S}_h$ the angular metric from x_0. Suppose that $(w_n x_0)_{n \in \mathbb{Z}}$ is a typical bi-infinite sample path of the random walk on $\pi_1(S)$ generated by μ. Let γ be the bi-infinite geodesic determined by the limit points of the sample path. Let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to γ in $\widetilde{S}_h$. The value of the height function $h_\gamma(t_n)$ is determined by the distance from the tangent vector at $\gamma(t_n)$ to the invariant laminations in $T^1(S_h)$.

Let θ_Λ be the constant from the definition of the height function, Definition 109. By Proposition 68, if $\gamma(t)$ is distance at most $r \leqslant \theta_\Lambda \leqslant 1$ in $T^1(S_h)$ from a leaf ℓ of one of the invariant laminations, then the endpoints of γ are distance at most $8r$ from the endpoints of ℓ in ∂S_h. We shall write $\overline{\Lambda}$ for the union of the extended invariant laminations, and $\partial \overline{\Lambda}$ for the boundary in $\partial \widetilde{S}_h \times \partial \widetilde{S}_h$. Then

$$\mathbb{P} \left(d_{T^1(S_h)}(\gamma^1(t_0), \overline{\Lambda}^1) \leqslant r \right) \leqslant \nu \times \check{\nu} \left(N_{8r}(\partial \overline{\Lambda}) \right),$$

and by Corollary 65 there are constants A_1 and $\beta > 0$ such that

$$\mathbb{P}\left(d_{T^1(S_h)}(\gamma^1(t_0), \overline{\Lambda}^1) \leqslant r \right) \leqslant A_1 r^\beta.$$

Action by the shift map then gives us the same result for all n.

Using the relation between distance r in $T^1(S_h)$ and the value of the height function R, there is a constant A_2 such that

$$\mathbb{P}(|h_\gamma(t_n)| \geqslant R) \leqslant A_2 e^{-\beta k^R}.$$

We shall choose the basepoint in $\widetilde{S}_h \times \mathbb{R}$ to be $\overline{x}_0 = \iota(x_0)$. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ. Let $\tau_\gamma(t_n)$ be the corresponding point on the test path, and let $\overline{\gamma}(p_n)$ be the nearest point projection of $\tau_\gamma(t_n)$ to $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$.

By Proposition 67, the distance between $\tau_\gamma(t_0)$ and $\tau_\gamma(t_n)$ grows linearly in n. As τ_γ is contained in an L-neighborhood of $\overline{\gamma}$, the distance between $\overline{\gamma}(p_0)$ and $\overline{\gamma}(p_n)$ also grows linearly in n.

Every point of $\overline{\gamma}([p_0, p_N]) \setminus N_R(S_0)$ is within distance $B_2 = B + 2L$ of a point $\overline{\gamma}(p_k)$, and the proportion of such points is at most $\mathbb{P}(|h_\gamma(t_0)| \geqslant R)$. Therefore

$$\lim_{N\to\infty} \frac{1}{|[p_0, p_N]|} |\overline{\gamma}([p_0, p_N]) \setminus N_R(S_0)| \leqslant \frac{B_2}{\ell_3} A_2 e^{-\beta k^R}.$$

As p_N tends to infinity as N tends to infinity, the result follows. □

Part II
Uniform Quasigeodesics

In this this section we prove the results about quasigeodesics that we need for the effective results in Part I.

We review in Section 5 the required properties of measured laminations and the Cannon-Thurston metric.

The proof that our test paths are quasi-geodesics is contained in Section 6.

In Section 6.1, we recall Farb's criteria [Far94] to show that a path in a hyperbolic space is a quasigeodesic, namely that the path is contained in a bounded neighborhood of a geodesic, and makes definite progress along the geodesic. The rest of Section 6 are the technical details required to verify the two properties for our test paths. We do so from two other properties, which we will now briefly describe below.

Recall that every non-exceptional geodesic γ in S_h determines a test path τ_γ in $\widetilde{S}_h \times \mathbb{R}$. We consider even sided regions in $\widetilde{S_h}$ whose boundary consists of arcs that alternate between the laminations. We call a region with four sides a rectangle. A *corner segment* is a segment $\gamma(I)$ of γ which is contained in an innermost rectangle, and which hits adjacent sides of the rectangle. We show that the test path $\tau_\gamma(I)$ over a corner segment is a *bottleneck*, so the geodesic in $\widetilde{S}_h \times \mathbb{R}$ connecting the endpoints of $\iota(\gamma)$ must pass close to $\tau_\gamma(I)$. We call segments of γ in between corner segments *straight segments*, and we show that the test path over a straight segment is quasigeodesic.

In Section 6.2 we give a precise definition of test paths we consider. In Section 6.3, we show that corner segments give rise to bottlenecks, and in Section 6.4 we show that straight segments are quasigeodesic. Finally, in Section 6.5 we prove Proposition 56, the additional property we need about the vertical projection from the image of the geodesic $\iota(\gamma)$ to the test path τ_γ.

5 Hyperbolic geometry and extended laminations

In this section, we briefly review some useful facts about hyperbolic metrics on surfaces, and define the notion of an extended lamination.

5.1 Close geodesics diverge exponentially

It is well known that if the lifts of two geodesics in $\mathrm{PSL}(2,\mathbb{R})$ are very close, then the distance between them increases exponentially as you move away from the closest point between them until they are a definite distance apart, and then grows linearly. More precisely, if they are distance θ apart, then the distance between them grows exponentially for a distance of length roughly $\log \frac{1}{\theta}$. We will use the following version of this result.

Proposition 69. *There are constants $\theta_0 > 0$ and $L_0 \geqslant 1$, such that for any two geodesics γ_1 and γ_2 in $\mathbb{H}^2$ whose lifts in $PSL(2,\mathbb{R})$ have closest points distance $\theta \leqslant \theta_0$ apart, if γ_1 has a unit speed parametrization $\gamma_1(t)$ with the closest point to γ_2 being $\gamma_1(0)$, then for all $|t| \leqslant \log \frac{1}{\theta}$,*

$$\tfrac{1}{L_0}\theta e^{|t|} \leqslant d_{PSL(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant L_0 \theta e^{|t|}. \tag{12}$$

Furthermore, the lower bound at $|t| = \log \frac{1}{\theta}$ holds for all t outside this range, i.e. for all $|t| \geqslant \log \frac{1}{\theta}$,

$$d_{PSL(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \geqslant \tfrac{1}{L_0}.$$

We give a detailed proof of this result in Appendix B for the convenience of the reader.

Definition 70. Suppose that ℓ and γ are geodesics with γ parametrized with unit speed. Suppose $\gamma(t)$ is the closest point of γ to ℓ and suppose that $d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1) = \theta$. We call the interval

$$E_\ell = [t - \log \tfrac{1}{\theta}, t_\ell + \log \tfrac{1}{\theta}]$$

the *exponential interval* for γ with respect to ℓ.

Note that E_ℓ is the interval on which the exponential estimate (12) holds.

5.2 Extended laminations

Given a full lamination on a hyperbolic surface, we define the resulting extended lamination as below.

Definition 71. Suppose that Λ is a measured lamination on a hyperbolic surface S_h, and suppose that all complementary regions of Λ are ideal polygons. We define the *extended lamination* $\overline{\Lambda}$ to be the union of Λ with additional leaves connecting every pair of ideal points for each ideal complementary region. We will call the additional leaves *extended leaves.*

If all ideal complementary regions are triangles, then the extended lamination equals the original lamination. If there are ideal polygons with more than three vertices, the extended lamination is not even a geodesic lamination, as there are intersecting leaves. Extended laminations are not minimal, as not every leaf is dense, but they are still closed. Although the extended lamination is not a lamination, it still makes sense to assign measures to transverse arcs, and if we assign the extended leaves measure zero, the resulting measure is the same as the measure from the original measured lamination. As the only way to obtain a finite measure is to make the extended leaves measure zero, the extended lamination is also not a geodesic current as there are transverse arcs of zero measure.

Proposition 72. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair measured laminations. Then the corresponding extended laminations $\overline{\Lambda}_+$ and $\overline{\Lambda}_-$ do not share any common leaves.*

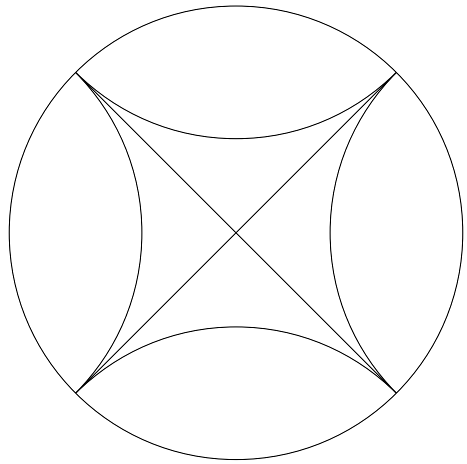

Figure 4: Extended leaves in a complementary region with four sides.

Proof. Suppose that the two extended laminations share a common leaf. The measured laminations do not share any common leaves, so the common leaf must be an extended leaf. But then the two laminations have ideal complementary regions that share a common point at infinity, contradicting Proposition (11.4). □

Proposition 73. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Then there is a constant $\alpha_\Lambda > 0$ such that any two leaves of the extended laminations $\overline{\Lambda}_+$ and $\overline{\Lambda}_-$ which intersect, intersect at angle at least α_Λ.*

Essentially the same proof as before works, but we give the details for the convenience of the reader.

Proof. Suppose that there is a sequence of pairs of intersecting leaves ℓ_n^-, ℓ_n^+, whose angles of intersection tend to zero. By compactness of S, we may pass to a convergent subsequence. As the extended laminations are closed, this limits to a pair of leaves with zero angle of intersection, so $\overline{\Lambda}_+$ and $\overline{\Lambda}_-$ share a common leaf, contradicting Proposition 72. □

Every geodesic γ in S_h has a unique lift in the unit tangent bundle $T^1(S_h)$.

Proposition 74. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Then there is a constant $\alpha_\Lambda > 0$ such that the distance in $T^1(S_h)$ between any two leaves of the extended laminations $\overline{\Lambda}_+^1$ and $\overline{\Lambda}_-^1$ is at least α_Λ.*

Proof. Suppose that there is a sequence of pairs of leaves ℓ_n^-, ℓ_n^+, whose lifts become arbitrarily close in $T^1(S_h)$. By compactness of S, we may pass to a convergent subsequence. As the extended laminations are closed, this limits to a pair of leaves which are tangent, and hence equal, implying that the extended laminations share a common leaf, contradicting Proposition 72. □

Finally, we record the fact that any sufficiently long segment of a leaf of the extended lamination intersects the other extended lamination. Essentially, the same argument as before works, but we give the details below.

Proposition 75. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Then there is a constant $\overline{L}_\Lambda > 0$ such that any leaf of either of the extended laminations $\overline{\Lambda}_+$ or $\overline{\Lambda}_-$ of length at least $\overline{L}_\Lambda$, intersects the other lamination.*

Proof. Breaking symmetry, suppose that the leaf belongs to $\overline{\Lambda}_+$. By swapping the laminations, the same argument works for $\overline{\Lambda}_-$.

By Proposition (11.3), there is a constant L_Λ such that every segment of a leaf of the invariant lamination Λ_+ of length at least L_Λ, intersects Λ_-. So it suffices to consider extended leaves. Let σ_n be a sequence of segments of extended leaves $\ell_n \in \overline{\Lambda}_+$, such that the σ_n are disjoint from Λ_-, and the length of the segments σ_n tends to infinity. As there are finitely many extended leaves for $\overline{\Lambda}_+$ in S_h, we may pass to a subsequence that converges to an infinite ray of an extended leaf of $\overline{\Lambda}_+$ such that the ray is disjoint from Λ_-. Note that such a ray is asymptotic to a boundary leaf ℓ of Λ_+. By Proposition (11.1) there is a lower bound α_Λ on the angle of intersection between the leaves of the two invariant laminations. It follows that an infinite subray of ℓ is also disjoint from Λ_-, contradicting Proposition (11.3). □

5.3 Complementary regions, polygons and cusps

In this section, we fix notation to describe subsets of either the surface S_h, or the universal cover $\widetilde{S}_h$, which have boundaries consisting of alternating arcs of the laminations Λ_+ and Λ_-.

Suppose that Λ is an invariant lamination. Then $S_h \setminus \Lambda$ has finitely many connected components, and each connected component lifts to an ideal polygon in $\widetilde{S}_h$ with its boundary consisting of leaves of Λ. We shall call these complementary regions *ideal polygons with boundary in* Λ, and there are countably many of these in the universal cover $\widetilde{S}_h$.

Suppose that R is an ideal polygon with boundary in Λ_+ (respectively, Λ_-). Suppose that ℓ is segment of a leaf of Λ_- (respectively, Λ_+) such that the endpoints of ℓ lie on adjacent sides of R. The adjacent sides meet in an ideal point of R and we call the component of $R \setminus \ell$ containing this ideal point a *cusp* of R. We say a cusp of R is *maximal*, if it is not contained in any larger cusp.

We now describe regions in $\widetilde{S}_h$ with boundary consisting of arcs alternately contained in Λ_+ and Λ_-. A *polygon* is a compact subset of $\widetilde{S}_h$ homeomorphic to a disc, whose boundary consists of an even number of arcs, alternately contained in Λ_+ and Λ_-. We (partially) organize polygons as follows. We call a polygon a *rectangle* if its boundary consists of four arcs. We call the polygon a *non-rectangular polygon* if it has more than four sides. We shall only consider polygons in $\widetilde{S}_h$ whose interiors embed in S_h.

The interior of a polygon may intersect other leaves of the laminations. If the interior of the polygon is disjoint from the leaves of Λ_+ and Λ_- then we shall call it an *innermost polygon.* An innermost polygon can be either an innermost rectangle, or an innermost non-rectangular polygon.

Definition 76. Let (S_h, Λ) be a hyperbolic metric on S together with a full pair of measured laminations. We say that the laminations are *suited* if every ideal complementary region R contains a unique non-rectangular polygon. In particular, any arc of the other invariant lamination that lies in R connects adjacent sides of R, and so determines a cusp.

The invariant laminations of a pseudo-Anosov map are suited; we include a proof below for convenience.

Proposition 77. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then the laminations are suited.*

Proof. By collapsing the complementary regions of an invariant lamination, say Λ_+, we obtain a a measured foliation F_+; see [Kap01, Chapter 11] for details. Since the laminations are uniquely ergodic, the resulting foliations are also uniquely ergodic and in particular, contain no saddle connections. Thus, the image of an ideal polygon with n sides is a singular leaf of F_+ with a single n-prong singularity. The measured foliation F_- (given by collapsing complementary regions of Λ_-) also has a singular leaf with an n-prong singularity at the same point. The pre-image of this leaf is an ideal polygon complementary to Λ_-. As the limit points of the two singular leaves alternate at the boundary at infinity, the two ideal polygons also have alternating ideal points at infinity, and so the intersection of the two ideal polygons is an innermost non-rectangular polygon P with $2n$ sides.

In particular, each side of the polygon P which intersects the interior of R determines a maximal cusp, and all other arcs of the other lamination which intersect R therefore also lie in cusps. Therefore these arcs have endpoints in adjacent sides of R, and so determine (non-maximal) cusps in R. □

Figure 5 shows two ideal quadrilaterals R_+ and R_- intersecting in an innermost polygon P with eight sides. The complement in each ideal quadrilateral of the innermost polygon P is a maximal cusp. All other arcs of Λ_+ intersecting R_- lie in maximal cusps, and so determine non-maximal cusps.

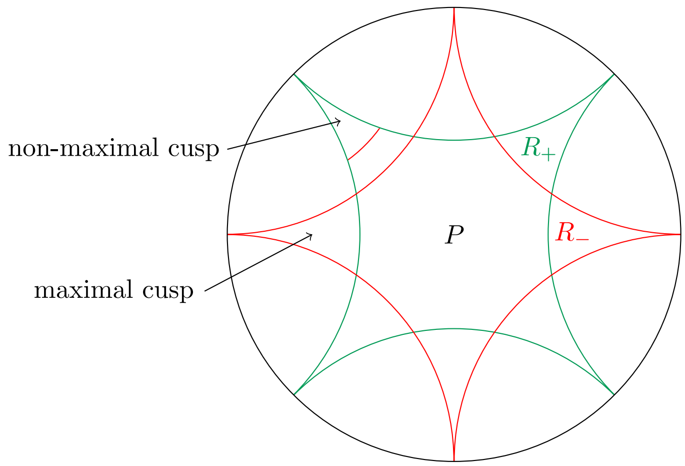

Figure 5: Two ideal polygons intersecting in a non-rectangular polygon.

Remark 78. It is possible to have a full pair of laminations Λ such that

- the pair is not suited, and yet
- the bi-infinite Teichmüller geodesic that they define gives a doubly degenerate hyperbolic 3-manifold with bounded geometry.

This can happen when one of the measured foliations has a saddle connection, and yet the bi-infinite Teichmüller geodesic lies in a thick part of Teichmüller space.

In light of Remark 78, our discussion in this section and its uses in the proofs of the effective theorems does not extend without extra hypothesis to doubly degenerate hyperbolic 3-manifolds with bounded geometry.

We observe below that there is an upper bound on the diameter of any innermost polygon.

Proposition 79. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant D_Λ such that diameter of any innermost polygon with boundary in $\Lambda_+ \cup \Lambda_-$ is at most D_Λ.*

Proof. By Proposition (11.3), there is an upper bound L_Λ on the length of any side of an innermost polygon. There are only finitely many non-rectangular innermost polygons in S_h, and the length of the boundary of a polygon is an upper bound on its diameter, so we may choose D_Λ to be nL_Λ, where n is the maximum number of sides of any innermost polygon with boundary in $\Lambda_+ \cup \Lambda_-$. □

Suppose that P is an innermost non-rectangular polygon. The intersection of the extended leaves of Λ_+ and Λ_- with P gives us a further finite collection of finite length geodesic segments in P which we call the *extended leaves* in P.

Proposition 80. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Suppose that P is an innermost non-rectangular complementary polygon. Then there is a* *constant θ_P such that if the lift of a non-exceptional geodesic (c.f. Definition 40) γ passes within distance θ_P in $T^1(S_h)$ of an extended leaf in P, then its distance in $T^1(S_h)$ is at least θ_P from all of the other extended leaves in P.*

Proof. If any two pre-images of geodesics in S_h intersect in $T^1(S_h)$, they are the same geodesic. Any innermost non-rectangular complementary polygon is compact and there are finitely many innermost non-rectangular polygons P_i. Hence, for each P_i there is a constant θ_{P_i} such that the θ_{P_i}-neighborhoods of all of the segments of the leaves in the polygon are disjoint in $T^1(S_h)$. By Proposition 79, the polygon has bounded diameter, and hence by Proposition 69, there is a constant θ'_{P_i} such that if a point on γ is within distance θ'_{P_i} of one of the leaves in the polygon, then it is within distance θ_{P_i} of that leaf for all t for which $\gamma(t)$ is in the polygon, and so is distance at least θ_{P_i} from all of the other leaves in the polygon. By choosing θ_P to be the minimum of θ_{P_i} we conclude the proof. □

5.4 Rectangles

In this section, we analyze rectangles and define their measures. Let R be a rectangle with opposite sides contained in leaves ℓ_1 and ℓ_2 of one of the invariant laminations. We say R is (ℓ_1, ℓ_2)-*maximal* if it is not contained in any larger rectangle with sides in ℓ_1 and ℓ_2. We will show that if two leaves ℓ_1 and ℓ_2 have a common leaf of intersection, then they bound a unique (ℓ_1, ℓ_2)-maximal rectangle, with upper and lower bounds on its measure.

Note that a rectangle is non-innermost if its interior intersects other leaves of the laminations.

Given a side of a rectangle in $\widetilde{S}_h$, we will distinguish between its hyperbolic and its Cannon–Thurston pseudometric length. We will call the pseudometric length the *measure* of the side, as it is defined in terms of the measured laminations. Note that every leaf (of the complementary lamination) that crosses the interior of a side of a rectangle also crosses the interior of the opposite side of that rectangle. It follows that opposite sides of a rectangle have equal measures, even though they may have different hyperbolic lengths. If the interior of a side does not cross any leaves of the complementary lamination then it has zero measure. A rectangle is a *square* if its sides have equal measures. We define the *measure* of a rectangle to be the product of the measures of two adjacent sides. By the definition, an innermost rectangle has measure zero.

We now fix some notation to refer to the side measures of a rectangle. Let α^+ be a side of the rectangle in Λ_+, and let α^- be a side of the rectangle in Λ_-. Define $dx(R) = \int_{\alpha^- \cap R} dx$ and $dy(R) = \int_{\alpha^+ \cap R} dy$. By the discussion above, these quantities do not depend on the choice of side. We define the measure of R to be $dx(R)dy(R)$. We say a non-innermost rectangle R has *positive measure* if $dx(R)dy(R) > 0$, and this will be the case if and only if its interior intersects leaves of both invariant laminations.

We specify below some notation for rectangles with positive measure, using the conventions in Figure 6.

Suppose that R is a rectangle with positive measure. The rectangle has two sides contained in leaves of Λ_+, which we shall label α^+ and β^+. Similarly, the rectangle has two sides contained in leaves of Λ_-, which we shall label α^- and β^-.

Since all sides of R have positive measure, the four leaves containing the sides of R have distinct endpoints at infinity (no two are asymptotic). They divide $\widetilde{S}_h$ into nine complementary regions, of which only R is compact. We call a (non-compact) complementary region in $\widetilde{S}_h \setminus R$ a *quadrant* if it meets exactly one corner of R. We call a pair of quadrants *opposite*, if they meet opposite corners of R. In Figure 6, the regions U and V form a pair of opposite quadrants.

We define the *optimal height* $z = z(R)$ of a positive measure rectangle R to be the value at which the sides of the rectangle $F_z(R)$ have equal measure, that is $F_z(R)$ is a square in $F_z(\widetilde{S}_h)$. So if the sides have measures $dx(R)$ and $dy(R)$, the optimal height is $\frac{1}{2}\log_k(dy(R)/dx(R))$, where k is the stretch factor of the pseudo-Anosov f, and the measure of each side at the optimal height is $\sqrt{dx(R)dy(R)}$. In particular, the optimal height for a square is at $z = 0$.

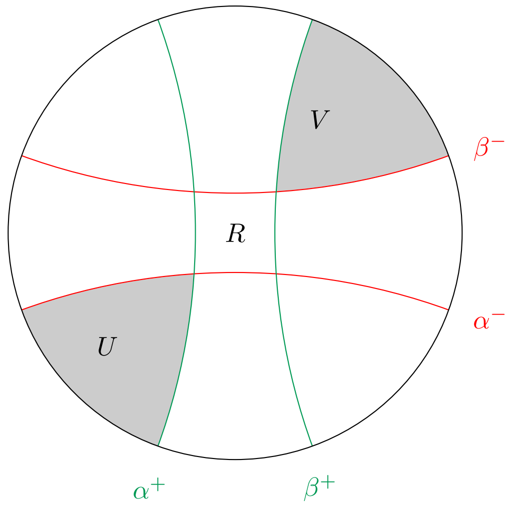

Figure 6: Opposite quadrants.

Suppose ℓ_1 and ℓ_2 are distinct leaves in Λ_-. We say that a leaf $\ell_+ \in \Lambda_+$ is a *common positive leaf* for ℓ_1 and ℓ_2 if

- ℓ_+ intersects both ℓ_1 and ℓ_2, and
- the arc of ℓ_+ between ℓ_1 and ℓ_2 has positive measure.

This is illustrated in Figure 7. Similarly, given distinct leaves ℓ_1 and ℓ_2 of Λ_+, we may define a leaf of Λ_- to be common positive leaf for ℓ_1 and ℓ_2 in the same way.

Since invariant laminations do not contain isolated leaves, by Proposition (11.1), the set of common positive leaves is a closed set.

The set of common positive leaves can be empty; for example, if no leaf of the complementary lamination intersects both ℓ_1 and ℓ_2, or if ℓ_1 and ℓ_2 are boundary leaves of an ideal complementary region in which case all complementary arcs between ℓ_1 and ℓ_2 have measure zero.

We say that two leaves ℓ_1 and ℓ_2 of the same lamination *bound* a rectangle, if the leaves contain opposite sides of a rectangle with positive measure. In particular, the set of common positive leaves is non-empty. Identifying ℓ_1 with $\mathbb{R}$, we conclude that the intersection points with ℓ_1 of common positive leaves is a closed bounded set. Since invariant laminations have no isolated leaves, it follows that this closed bounded set contains no isolated points. We call the common positive leaves that give the extrema of this set the *outermost* common positive leaves. It also follows that the arc of ℓ_1 (similarly of ℓ_2) between the outermost common positive leaves has positive measure. The arcs of the outermost common positive leaves together with the arcs of ℓ_1 and ℓ_2 that they determine, are the sides of an (ℓ_1, ℓ_2)-*maximal* rectangle, that is, the rectangle is not contained in a larger rectangle with sides in ℓ_1 and ℓ_2.

Suppose that R is an (ℓ_1, ℓ_2)-maximal rectangle. Then the two sides of R not contained in ℓ_1 or ℓ_2, being segments of the outermost common leaves, are contained in boundary leaves of the other lamination. Maximality of R implies that these sides contain sides of a non-rectangular polygon. The sides of R in ℓ_1 and ℓ_2 need not contain sides of an innermost non-rectangular polygon, and so R may be contained in a larger rectangle, which at least one of ℓ_1 and ℓ_2 intersect in its interior.

We now show that there is a lower bound on the measure of an (ℓ_1, ℓ_2) maximal rectangle.

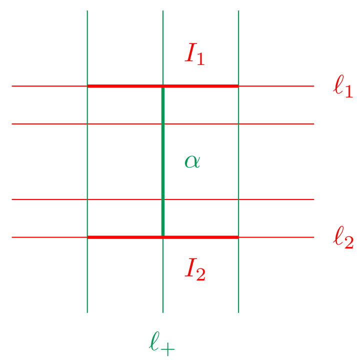

Figure 7: Leaves of Λ_- intersecting a common leaf of Λ_+.

Proposition 81. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant $A_\Lambda > 0$, such that any two leaves ℓ_1 and ℓ_2 of Λ_+ that have a common positive leaf in Λ_- bound an (ℓ_1, ℓ_2)-maximal rectangle of measure at least A_Λ. Similarly, any two leaves ℓ_1 and ℓ_2 of Λ_- that have a common positive leaf in Λ_+ bound a maximal rectangle of measure at least A_Λ.*

Proof. Suppose that there is a sequence of pairs of leaves ℓ_n, ℓ'_n containing opposite sides α_n and α'_n of an (ℓ_n, ℓ'_n)-maximal rectangle R_n, so that the measure of the (ℓ_n, ℓ'_n)-maximal rectangle R_n tends to zero. Let β_n and β'_n be the other sides of the (ℓ_n, ℓ'_n)-maximal rectangle. As the rectangle is (ℓ_n, ℓ'_n)-maximal, each side β_n (respectively β'_n) contains a side of an innermost non-rectangular polygon P_n (respectively P'_n) in $\widetilde{S}_h \setminus (\Lambda_+ \cup \Lambda_-)$, whose interior is disjoint from R_n.

As there are only finitely non-rectangular innermost polygons in S_h, we may pass to a subsequence where the innermost polygons at each end do not change. We denote these non-rectangular polygons P and P'. By compactness of S_h, we may pass to a subsequence of rectangles which converge to a (possibly degenerate) rectangle R of measure zero. A measure zero rectangle is either an innermost rectangle, or a degenerate rectangle given by a subinterval of a leaf, or a point. In all cases, at least one of the two leaves ℓ and ℓ' that arise as limits of ℓ_n and ℓ'_n respectively, is a boundary leaf of both P and P'. Breaking symmetry, suppose ℓ is a boundary leaf of both P and P'. Orienting ℓ, suppose that P and P' lie on opposite sides of ℓ. Then ℓ cannot be a limit of other leaves from either side and is hence isolated, a contradiction. Suppose then that P and P' lie on the same side of ℓ. Then P and P' are contained in a single ideal complementary region of the lamination containing ℓ, a contradiction to the fact that the pair of laminations is suited. □

We finally record one more useful fact that there is an upper bound on the measure of any rectangle.

Proposition 82. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a pair of suited measured laminations with bounded geometry. Then there is a constant $A_{\max}$ such that any rectangle R with sides in the invariant laminations has measure at most $A_{\max}$.*

Proof. The vertical flow is measure preserving for rectangles. Let R_z be the image of R under the vertical flow at optimal height z, at which height R_z is a square, in the intrinsic Cannon-Thurston metric on $\widetilde{S}_h \times \{z\}$.

As the pair of laminations has bounded geometry, the Teichmüller geodesic determined by the pair of laminations is contained in a compact subset of moduli space. In particular, there are constants Q and c such that for all z, the intrinsic Cannon-Thurston metric on $\widetilde{S}_h \times \{z\}$ is (Q, c)-quasi-isometric to $\widetilde{S}_h$.

Every point in $\widetilde{S}_h$ is a bounded distance from an innermost non-rectangular region, hence every point in $\widetilde{S}_h \times \{z\}$ is a bounded distance from a non-rectangular region. As an innermost non-rectangular region cannot be contained in a square, there is an upper bound on the diameter of any square, and hence the measure of the square. □

5.5 Corner segments and straight segments

In this section we define some notation that will be useful for describing specific subsegments of geodesics.

Definition 83. Let γ be a geodesic in (S_h, Λ). We say a compact interval I is a *corner segment* if the segment $\gamma(I)$

- is properly embedded in an innermost rectangle, and
- the endpoints of $\gamma(I)$ lies in different laminations.

We include the degenerate case in which I is a point and $\gamma(I)$ a vertex of an innermost rectangle.

Corner segments can occur when a geodesic enters or exits an ideal complementary region through one of its cusps.

Definition 84. Let γ be a geodesic in (S_h, Λ). We say a compact interval I is a *straight segment*, if it contains exactly two corner segments, each one adjacent to an endpoint of I.

We now show that for a suited pair of laminations there are exactly two types of straight segments. Recall that if (S_h, Λ) is suited, then every non-rectangular polygon P is the intersection $P = R_+ \cap R_-$, where R_+ and R_- are ideal complementary regions to Λ_+ and Λ_- respectively, and each region of $R_+ \setminus P$ and $R_- \setminus P$ is a cusp.

Proposition 85. *Let (S_h, Λ) be a hyperbolic metric on S together with a suited pair of measured laminations, and let γ be a non-exceptional geodesic in S_h. Then every corner segment is contained in a straight segment, and furthermore, every straight segment consists of either the intersection of γ with a single ideal complementary region, or the intersection of γ with the union of two ideal complementary regions intersecting in a non-rectangular polygon.*

Proof. As γ is non-exceptional, the intersection of γ with any ideal complementary region R is a compact subinterval. Suppose that $\gamma(I) = \gamma \cap R$ has both endpoints in cusps, i.e. neither endpoint lies in the non-rectangular polygon. Then both endpoints lie in rectangles, and hence in corner segments. Thus, $\gamma(I)$ is a straight segment contained in a single ideal complementary region.

Now suppose $\gamma(I) = \gamma \cap R$ has both endpoints in the non-rectangular polygon $P = R \cap R'$ contained in R, where R' is an ideal complementary region of the other lamination. It follows that

- $\gamma(I)$ is contained in $\gamma(I') = \gamma \cap R'$, and
- both endpoints of $\gamma(I')$ are in cusps of R'.

Thus $\gamma(I)$ is again a straight segment contained in a single ideal complementary region.

We may now suppose that exactly one endpoint of $\gamma(I) = \gamma \cap R$ is contained in the non-rectangular polygon $P = R \cap R'$. Then $\gamma(I') = \gamma \cap R'$ contains the other endpoint of $\gamma \cap P$ and so $\gamma(I \cup I')$ is a straight segment with endpoints in cusps of R and R'.

□

Finally, we show that the intersection of γ and an innermost polygon is contained in a straight segment.

Proposition 86. *Let (S_h, Λ) be a hyperbolic metric on S together with a suited pair of measured laminations, and let γ be a non-exceptional geodesic in S_h. Then every intersection segment $\gamma \cap R$ with an innermost polygon P is contained in a straight segment. In particular, straight segments are dense in γ.*

Proof. Suppose that $P = R_+ \cap R_-$ is an innermost polygon, where R_+ and R_- are ideal complementary regions of Λ_+ and Λ_-.

Suppose that P is a rectangle. If $\gamma \cap P$ is a corner segment, then we are done by Proposition 85. So suppose that γ intersects opposite sides of the rectangle P. Breaking symmetry, assume that the endpoints of $\gamma \cap P$ are contained in R_+. Each side of P in R_+ bounds a cusp in R_-, with one of the cusps contained in the other. The geodesic γ therefore enters the smaller cusp C. Let P' be the last innermost rectangle in C that $\gamma \cap C$ intersects. Then $\gamma \cap P'$ is a corner segment. Therefore $\gamma \cap P$ is contained in a segment $\gamma \cap R_-$ which terminates in a corner segment at one end, and so is a subset of a straight segment by Proposition 85.

Now suppose that P is not a rectangle. Each region of $R_+ \setminus P$ and $R_- \setminus P$ is a cusp. If both endpoints of $\gamma \cap P$ lie in the same lamination, say R_+, then $\gamma \cap R_-$ has both endpoints in cusps of R_-. Thus, $\gamma \cap P$ is contained in the straight segment $\gamma \cap R_-$. If both endpoints of $\gamma \cap P$ lie in different laminations, then one endpoint of $\gamma \cap (R_+ \cup R_-)$ lies in a cusp of $R_+ \setminus P$, and the other endpoint lies in a cusp of $R_- \setminus P$. Again, $\gamma \cap P$ is contained in the straight segment $\gamma \cap (R_+ \cup R_-)$.

As innermost regions are dense in S_h, and γ is non-exceptional, intersections of innermost regions are dense in γ. As every intersection with an innermost region is contained in a straight segment, straight segments are dense in γ, as required. □

5.6 Bottlenecks

The main result of this section is that a non-innermost rectangle in $(\widetilde{S}_h, \Lambda)$ creates a *bottleneck* in $\widetilde{S}_h \times \mathbb{R}$, which we now define.

Definition 87. Let X be a geodesic metric space, and let U and V be subsets of X. A set $R \subset X$ is an (r, K)-bottleneck for U and V if the distance from U to V is at least r, and any geodesic from U to V passes within distance K of R.

We will show that an optimal height rectangle $F_z(R)$ is an (r, K)-*bottleneck* with respect to either pair of opposite quadrants. The constants r and K depend on the measure of the rectangle (as well as various constants depending on the pseudo-Anosov map f), and as the measure of the rectangle tends to zero, r tends to zero and K tends to infinity.

Lemma 88. *(Rectangles create bottlenecks.) Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Let R be a rectangle of measure at least $A > 0$ and optimal height z. Then there are constants $r > 0$ and $K \geqslant 0$ (that depend on f and A) such that the optimal height rectangle $F_z(R) = R \times \{z\}$ is an (r, K)-bottleneck for the flow sets $F(U)$ and $F(V)$ over any pair of opposite quadrants U and V of R.*

We start by showing that there is a lower bound on the distance between the suspension flow sets over opposite quadrants of a transverse rectangle R, in terms of the measure of R.

Proposition 89. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. For any $A > 0$ there is $r > 0$ such that for any rectangle of measure at least A, the suspension flow sets over opposite quadrants are Cannon-Thurston distance at least r apart in $\widetilde{S}_h \times \mathbb{R}$.*

Proof. Suppose that U and V are opposite quadrants, as illustrated in Figure 6, so U the region bounded by α^+ and α^-, and V the region bounded by β^+ and β^-. Let the corresponding suspension flow sets over these regions be $F(U)$ and $F(V)$. Suppose that the measures of the sides of the rectangle R are $dx(R) = a > 0$ and $dy(R) = b > 0$. The optimal height of R is $z = \frac{1}{2}\log_k(b/a)$, and so the measure of the diagonal (from the corner of R meeting U to the corner of R meeting at V) at this height is $\sqrt{2ab}$.

We now give a lower bound for the distance in $\widetilde{S}_h \times \mathbb{R}$ between $F(U)$ and $F(V)$. Let γ be a shortest path in $\widetilde{S}_h \times \mathbb{R}$ from $F(U)$ to $F(V)$. By definition, $\int_\gamma dx \geqslant a$ and $\int_\gamma dy \geqslant b$. Let z_+ and z_- be the largest and smallest height attained along γ; these exist because γ is compact. The Cannon-Thurston pseudo-metric length of the diagonal of R at the optimal height is an upper bound on the length of γ, so the length of γ is at most $\sqrt{2ab}$. We deduce that the difference in heights between any two points on γ is at most $\sqrt{2ab}$, in particular $z_+ - z_- \leqslant \sqrt{2ab}$. The Cannon-Thurston distance in $\widetilde{S}_h \times \mathbb{R}$ between α^+ and β^+ at any height $z \leqslant z_+$ is at least ak^{z_-}. Similarly, the Cannon-Thurston distance between α^- and β^- at any height $z \geqslant z_-$ is at least bk^{-z_+}. Therefore the length of γ is at least $ak^{z_-} + bk^{-z_+}$. If we set z_0 to be the average of z_+ and z_-, i.e. $z_0 = \frac{1}{2}(z_+ + z_-)$, then

$$z_+ \leqslant z_0 + \tfrac{1}{2}\sqrt{2ab} \quad \text{and} \quad z_- \geqslant z_0 - \tfrac{1}{2}\sqrt{2ab}.$$

In particular,

$$ak^{z_-} + bk^{-z_+} \geqslant ak^{z_0 - \sqrt{ab/2}} + bk^{-z_0 - \sqrt{ab/2}},$$

which may be rewritten as

$$ak^{z_-} + bk^{-z_+} \geqslant k^{-\sqrt{ab/2}} \left(ak^{z_0} + bk^{-z_0} \right).$$

The right hand side is minimized when $z_0 = \frac{1}{2}\log_k(b/a)$, so the length of α is at least $r = k^{-\sqrt{ab/2}} 2\sqrt{ab}$, which only depends on the measure $ab \geqslant A$ of the rectangle R. □

Proposition 90. *Suppose that (X, d) is a δ-hyperbolic metric space. Suppose that U and V are convex sets in X that are distance $r \geqslant 0$ apart. Then any geodesic from U to V is contained in $N_{2\delta+r}(U) \cup N_{2\delta+r}(V)$ and intersects $N_{2\delta+r}(U) \cap N_{2\delta+r}(V)$.*

Proof. Suppose that $\eta = [a, b]$ is a geodesic of length $r + \epsilon$, where $a \in U$ and $b \in V$. Suppose u is a point of U and v a point of V and $\gamma = [u, v]$ a geodesic from u to v. By the thin triangles property, γ is contained in a 2δ-neighborhood of $[u, a] \cup [a, b] \cup [b, v]$, and hence in a 2δ-neighborhood of $N_r(U) \cup N_r(V)$.

The geodesic η itself is contained in $N_r(U) \cap N_r(V)$. If γ passes within distance 2δ of η, then γ intersects $N_{2\delta+r}(U) \cap N_{2\delta+r}(V)$, as required. Otherwise, γ is contained in a 2δ-neighborhood of $[u, a] \cup [b, v]$, and hence in a 2δ-neighborhood of $U \cup V$. In particular, there is a point on γ which is distance at most 2δ from both U and V, so γ again intersects $N_{2\delta+r}(U) \cap N_{2\delta+r}(V)$, as required. □

Proposition 91. *Suppose $\theta > 0$ and $r > 0$ are constants. Then there is a constant $K > 0$ such that for any two geodesics γ_1 and γ_2 in $\mathbb{H}^2$ meeting at a point x with an angle at least θ, we have $N_r(\gamma_1) \cap N_r(\gamma_2) \subseteq N_K(x)$ in the hyperbolic metric.*

Proof. Suppose that y is a point distance at most r from both γ_1 and γ_2. Let p_1 and p_2 be the respective points on γ_1 and γ_2 closest to y. Then x, y, p_1 and x, y, p_2 form two right angled triangles, with angles θ_1 and θ_2 at x such that $\theta_1 + \theta_2 = \theta$. This is illustrated in Figure 8.

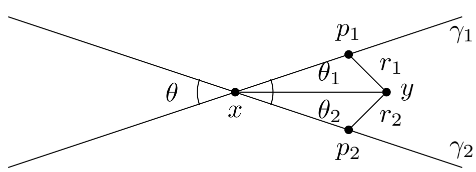

Figure 8: Intersecting geodesics.

In particular, $\min\{\theta_1, \theta_2\} \geqslant \frac{1}{2}\theta$, and up to relabeling, we may assume that $\theta_1 \geqslant \frac{1}{2}\theta$. Let $d = d(x, y)$. Using the sin rule for right angled triangles in $\mathbb{H}^2$,

$$\sin\theta_1 = \frac{\sinh r_1}{\sinh d}.$$

which we may rewrite as

$$\sinh d = \frac{\sinh r_1}{\sin \theta_1}.$$

We will use the following elementary estimates: for $x \leqslant \frac{\pi}{2}$, $\sin x \geqslant \frac{1}{2}x$ and $\sinh x \leqslant \frac{1}{2}e^x$. Together with $\theta_1 \geqslant \frac{1}{2}\theta$ and $r_1 \leqslant r$, we deduce

$$\sinh d \leqslant \tfrac{1}{\theta}e^r.$$

We may therefore choose $K = \sinh^{-1}(e^r/\theta)$ to conclude the proof. □

Suppose that R is a rectangle with optimal height z. We now show that for opposite quadrants U and V for R, the intersection of the metric regular neighborhoods of $F_z(U)$ and $F_z(V)$ with the fiber $\widetilde{S}_h \times \{z\}$ are contained in a bounded neighborhood of the optimal height square $F_z(R)$.

Proposition 92. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. For any positive constants $A > 0$ and $r > 0$, there is a positive constant $K > 0$ such that for any opposite quadrants U and V of a rectangle R with measure at least A and optimal height z*

$$N_r(F_z(U)) \cap N_r(F_z(V)) \subset N_K(F_z(R)).$$

in the hyperbolic metric on $\widetilde{S}_h \times \{z\}$.

Proof. We may assume that $z = 0$ and that R is a square with the measures of the sides satisfying $a \geqslant \sqrt{A}$. We will choose K to be the constant from Proposition 91, with the given choice of r, and θ chosen to be α_Λ, the minimal angle of intersection of any two leaves from each invariant lamination, from Proposition (11.1).

Let x be a point that is distance at most r from both U and V. If x lies in R, there is nothing to prove. So we may assume that x does not lie in R. We will then show that x is distance at most r from each pair of intersecting geodesics containing the sides of R. As these geodesics intersect in the corners of R, the result follows from Proposition 91. We now give the details.

Suppose x lies in U. We refer to Figure 6. Since any geodesic from x to V crosses both α^+ and α^-, the point x lies within distance r of both α^+ and α^-. These two geodesics meet at angle at least α_Λ, by Proposition (11.1), so by Proposition 91, x is within distance K of their intersection point, which is one of the corners of R.

The same argument works if x lies in V, so we may now assume that x does not lie in either U or V, and breaking symmetry, we may assume that x lies in one of the regions W_1, W_2 and W_3 in Figure 6. We consider each case in turn.

If x lies in the region W_1, then x is distance at most r from both α^+ and α^-, which intersect. Similarly, if x lies in the region W_2, then x is distance at most r from both α^+ and β^-, which intersect. Finally, if x lies in the region W_3, then x is distance at most r from both β^+ and β^-, which intersect. In all three cases, by Proposition (11.1), the angles of intersection of the leaves is at least α_Λ. Hence, Proposition 91 applies and x is contained in a K-neighborhood of the intersection points of the pairs of geodesics. As the intersection points are corners of the rectangle R, the result follows. □

By the quasi-isometry between the hyperbolic and Cannon–Thurston metric, we deduce Proposition 92 also in the Cannon–Thurston metric. We now show that any geodesic in $\widetilde{S}_h \times \mathbb{R}$ between the suspension flow sets of opposite quadrants is contained in a bounded neighborhood of the suspension flow sets, and passes within a bounded neighborhood of the optimal height rectangle.

Proposition 93. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. For any positive constant $A > 0$ there is a positive constant $K > 0$ such that if γ is any Cannon–Thurston geodesic in $\widetilde{S}_h \times \mathbb{R}$ intersecting both suspension flow sets $F(U)$ and $F(V)$ of opposite quadrants of a rectangle R with measure at least A and optimal height z, then γ is contained in $N_K(F(U) \cup F(V))$, and intersects $N_K(F_z(R))$.*

Proof. By Proposition 89, given $A > 0$, there is a constant $r > 0$ such that the suspension flow sets $F(U)$ and $F(V)$ over opposite quadrants U and V are Cannon-Thurston distance at least r apart. Since $F(U)$ and $F(V)$ are convex, by Proposition 90 any geodesic from $F(U)$ to $F(V)$ is contained in $N_{2\delta+r}(F(U) \cup F(V))$, and passes through $N_{2\delta+r}(F(U)) \cap N_{2\delta+r}(F(V))$. It therefore suffices to prove that there is a $K > 0$ such that $N_{2\delta+r}(F(U)) \cap N_{2\delta+r}(F(V))$ is contained in $N_K(F_z(R))$.

We may now assume that the optimal height is $z = 0$, at which height R is a square of side lengths $a \geqslant \sqrt{A}$. Let (p, z) be a point in $\widetilde{S}_h \times \mathbb{R}$ that is Cannon-Thurston distance at most $2\delta + r$ from both $F(U)$ and $F(V)$. A path starting at (p, z) with length at most $2\delta + r$, when projected into $S_h \times \{z\}$ along suspension flow lines, changes its length by a factor of at most $k^{2\delta+r}$. So in the Cannon-Thurston metric on $S_h \times \{z\}$, the point (p, z) is distance at most $r_1 = (2\delta + r)k^{2\delta+r}$ from both $F_z(U)$ and $F_z(V)$. In the intrinsic Cannon-Thurston pseudometric on $S_h \times \{z\}$, the distance from U to V is at least $\max\{ak^z, ak^{-z}\}$. This implies that $\max\{ak^z, ak^{-z}\} \leqslant 2r_1$, and thus $|z| \leqslant r_2 = \log_k(2r_1) - \log_k(a)$. Projecting down to $z = 0$ implies that $(p, 0)$ is distance at most $r_3 = r_1 k^{r_2}$ from both $F_0(U)$ and $F_0(V)$.

By Proposition 38, the point $(p, 0)$ lies hyperbolic distance at most $r_4 = Q_\Lambda r_3 + c_\Lambda$ from both $F_0(U)$ and $F_0(V)$. Let K_1 be the constant from Proposition 92, with r chosen to be r_4, and the choice of A from the initial assumption above. Proposition 92 now implies that as $(p, 0)$ lies in $N_{r_4}(F_0(U)) \cap N_{r_4}(F_0(V))$, the point $(p, 0)$ lies in $N_{K_1}(F_0(R))$. As $|z| \leqslant r_2$, the point (p, z) lies in a $(K_1 + r_2)$-neighborhood of $F_0(R)$. So we may choose $K = \max\{2\delta + r, K_1 + r_2\}$, which only depends on A, f and S_h, as required. □

6 Quasigeodesics in the Cannon-Thurston metric

In this section, from a non-exceptional geodesic γ in $\widetilde{S}_h$, we construct an explicit quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$ whose limit points are the images of the limit points of γ by the Cannon-Thurston map.

Recall that, by definition, the Cannon–Thurston images of the limit points of a non-exceptional geodesic γ in $\widetilde{S}_h$ are distinct and hence determine a geodesic $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, which we shall call the *target geodesic*. As the ladder $F(\gamma)$ of γ is quasiconvex, the target geodesic $\overline{\gamma}$ is contained in a bounded neighborhood of $F(\gamma)$. In particular, the nearest point projection of $\overline{\gamma}$ to $F(\gamma)$ is quasigeodesic. In particular, there is a function $h_\gamma(t)$ such that the path $(\gamma(t), h_\gamma(t))$ is quasigeodesic.

In this section, we will give an explicit formula for h_γ. In fact, $h_\gamma(t)$ will be defined in terms of the unit tangent vector to $\gamma(t)$. We will define a function $h\colon T^1(S_h) \setminus \overline{\Lambda}^1 \to \mathbb{R}$, and then define $h_\gamma(t) = h(\gamma^1(t))$. We will call the path $\tau_\gamma(t) = (\gamma(t), h_\gamma(t))$ in $F(\gamma)$ the *test path* associated to γ. We call the function $h_\gamma(t)$ the *height function* for γ. We will show that the test path is an (unparametrized) quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$ connecting the limit points of $\iota(\gamma)$, and we emphasize that the unit speed parametrization of γ in $\mathbb{H}^2$ does not give a quasigeodesic parametrization of the test path.

We now specify the height function we will use. We shall write $\log_k$ for the logarithm function with base k, where $k = k_f > 1$ is the parameter from the definition of the Cannon-Thurston metric, which is the stretch factor of f in the case that the laminations are the invariant laminations of a pseudo-Anosov map f.

Definition 94. Let (S_h, Λ) be a hyperbolic metric, together with a regular pair of measured laminations. Let $\overline{\Lambda}^1_-$ and $\overline{\Lambda}^1_+$ be the lifts in $T^1(S_h)$ of the extended laminations determined by Λ, and let $\theta_\Lambda > 0$ be a positive constant. We define the *height function* $h_{\theta_\Lambda}\colon T^1(S_h) \setminus (\overline{\Lambda}^1_+ \cup \overline{\Lambda}^1_-) \to \mathbb{R}$ to be

$$h_\theta(v) = \log_k \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(v, \overline{\Lambda}^1_+)} - \log \frac{1}{\theta_\Lambda} \right\rfloor_1 - \log_k \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(v, \overline{\Lambda}^1_-)} - \log \frac{1}{\theta_\Lambda} \right\rfloor_1$$

Here $\lfloor x \rfloor_c = \max\{x, c\}$ is the standard floor function. As the two extended laminations are a positive distance apart in $T^1(S_h)$, for sufficiently small θ_Λ, at most one of the terms on the right hand side above will be non-zero.

We prove below that for a sufficiently small choice of θ_Λ, the test path determined by the corresponding height function is quasigeodesic.

Theorem 95. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants $\theta_\Lambda > 0$, $Q \geqslant 1$ and $c \geqslant 0$, such that for any non-exceptional geodesic γ in S_h, with a unit speed parametrization $\gamma(t)$, the test path $\tau_\gamma(t) = (\gamma(t), h_{\theta_\Lambda}(\gamma^1(t)))$ is an unparametrized (Q, c)-quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$ with the same limit points as $\iota(\gamma)$, where h_{θ_Λ} is the height function from Definition 54.*

As we shall fix a sufficiently small constant θ_Λ, depending only on (S_h, Λ), we shall just write h for h_{θ_Λ}. See Section 6.2.2 for the exact choice of θ_Λ that we use. Furthermore, we will write $h_\gamma(t)$ for $h_{\theta_\Lambda}(\gamma^1(t))$.

Here is the basic intuition behind Theorem 95. We partition the geodesic $\gamma(t)$ into arcs where over each arc either the geodesic is far from both laminations or it comes close to one of the laminations. If it comes r-close to a lamination, say Λ_+ then there is a fellow travel of length $\log \frac{1}{r}$ with a leaf ℓ of Λ_+. This means that in $\widetilde{S}_h \times \mathbb{R}$ there is a short cut through the ladder (hyperbolic plane) of suspension flow lines through ℓ which has maximum height $\log_k(\log \frac{1}{r})$.

This intuition works well when the geodesic has a large intersection with a complementary region that is an ideal triangle. There are two main technical issues to address.

Firstly, there may be complementary ideal regions with more than three sides. Suppose that a geodesic crosses a non-triangular ideal complementary region entering and leaving the region through non-adjacent cusps and with a very small angle. As the geodesic traverses through the middle of the complementary region, it is far from all boundary leaves, and yet the height of the test path in $\widetilde{S}_h \times \mathbb{R}$ should be large. In this portion the geodesic is close to an extended leaf and so we overcome the difficulty by modifying the distance functions to consider distances to the extended laminations.

Secondly, there can be long geodesic segments that are close to leaves of one of the laminations, but do not spend very long in any single complementary region. In this case, there may be no simple shortcut through a single leaf, and yet we show that the $\log_k(\log \frac{1}{r})$ height function gives a quasigeodesic.

6.1 The test path is quasigeodesic

We will use the following criteria due to Farb [Far94] (see also Hoffoss [Hof07, page 216]) to ensure that a path is a quasigeodesic.

Definition 96. Suppose that (X, d) is a metric space. Suppose that τ is a path in X with unit speed parametrization and suppose that γ is a geodesic in X. We say that τ *makes (L, M)-uniform progress along* γ if there is a constant L such that for any two points distance at least L apart along τ, the distance between their nearest point projections to γ is at least M.

Theorem 97. [Far94][Hof07, page 216] *Suppose that τ is parametrized unit speed path in $\mathbb{H}^3$ such that*

- *τ is contained in a bounded neighborhood of a geodesic γ, and*
- *there is a constant $L > 0$ such that τ makes $(L, 1)$-uniform progress along γ.*

Then τ is a quasigeodesic.

As the Cannon-Thurston pseudo-metric is quasi-isometric to the hyperbolic metric on $\mathbb{H}^3$, it suffices to show the following two results.

Lemma 98. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant K such that for any non-exceptional geodesic γ in $\mathbb{H}^2$, the corresponding test path τ_γ is contained in a K-neighborhood of $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$.*

Let Q_M and c_M be the quasi-isometry constants between $\widetilde{S}_h \times \mathbb{R}$ and $\mathbb{H}^3$. Then $(L, 1)$-uniform progress in $\mathbb{H}^3$ will be implied by (L', M)-uniform progress in $\widetilde{S}_h \times \mathbb{R}$, where $M = Q_M + c_M$. In fact, assuming Lemma 98, any point on τ_γ is distance at most K from $\overline{\gamma}$. Therefore, it suffices to show that there is a constant L, such that any pair of points distance at least L apart along τ_γ are distance at least $M + 2K$ apart in $\widetilde{S}_h \times \mathbb{R}$.

Lemma 99. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then for any constant M there is a constant L such that for any non-exceptional geodesic γ, and any two points p and q distance at least L apart along the test path τ_γ, the distance in $\widetilde{S}_h \times \mathbb{R}$ between p and q is at least M.*

There are two key tools which we will use to show that these conditions hold.

(1) *Tame bottlenecks:* Test path segments over corner segments are tame bottlenecks.

   Recall that an interval I is a *corner segment* if $\gamma(I)$ is embedded in an innermost rectangle and the endpoints of $\gamma(I)$ lie in different laminations. We prove that the test path segment $\tau_\gamma(I)$ over $\gamma(I)$ is a bottleneck for the flow sets over the two complementary components of $\gamma(\mathbb{R} \setminus I)$. We also show that the bottleneck is *tame*, that is, $\tau_\gamma(I)$ has bounded length.

(2) *Straight intervals:* Test path segments over straight intervals are quasigeodesic.

   Recall that an interval I is *straight* if $\gamma(I)$ contains exactly two corner segments, each one adjacent to an endpoint of I. We prove that the test paths segments over straight intervals are quasigeodesic. There are exactly two cases: either $\gamma(I)$ is the intersection of γ with a single ideal complementary region, or the intersection with two complementary regions intersecting in a non-rectangular polygon.

We now state precise versions of the results described in the two items above, and two additional properties of the construction we will use. In the next two sections we use these results to deduce that the test path is quasigeodesic, by showing how Farb's criterion, namely Theorem 97 applies.

We first state the tame bottlenecks result. We shall prove it in Section 6.3.

Lemma 100. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants $r > 0$ and $K \geqslant 0$, such that for any corner segment $I = [t_1, t_2]$ of any non-exceptional geodesic γ, there are parameters $u \leqslant t_1 \leqslant t_2 \leqslant v$ such that the set $\tau_\gamma([u, v])$ is an (r, K)-bottleneck for the ladders over $\gamma((-\infty, u])$ and $\gamma([v, \infty))$. Furthermore, the length of $\tau_\gamma([u, v])$ is at most K.*

We now state the result for test paths over straight intervals. We shall prove this in Section 6.4.

Lemma 101. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants Q and c such that for any non-exceptional geodesic γ, and any straight interval I, the test path $\tau_\gamma(I)$ is (Q, c)-quasigeodesic.*

6.1.1 The test path is close to a geodesic

In this section, we use Lemma 100 and Lemma 101 to show that the test path τ_γ lies in a bounded neighborhood of the geodesic $\overline{\gamma}$, verifying the first condition in Theorem 97.

Lemma 98. Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant $K > 0$ such that for any non-exceptional geodesic γ in S_h, and for any geodesic $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$ connecting the limit points of $\iota(\gamma)$, the corresponding test path τ_γ is contained in a K-neighborhood of $\overline{\gamma}$, i.e.

$$\tau_\gamma \subseteq N_K(\overline{\gamma}).$$

This result is obtained as follows. By Lemma 100, the test path over any corner segment of γ is a bounded distance from the geodesic $\overline{\gamma}$. By Proposition 85, every straight segment has both endpoints a bounded distance from the geodesic $\overline{\gamma}$. Stability of quasigeodesics then implies that the entire straight segment is contained in a bounded neighborhood of $\overline{\gamma}$.

We now give the details for the proof of Lemma 98.

Proof (of Lemma 98). As γ is non-exceptional, its intersection with each ideal complementary region has finite diameter. The ideal complementary regions are dense in $\widetilde{S}_h$, and hence such intersection segments are dense in γ. By Proposition 85, each such intersection segment of γ is contained in a straight segment.

By Lemma 100, there is a constant K_1 such that if $\gamma(t)$ is in a corner segment, then $\tau_\gamma(t)$ is distance at most K_1 from $\overline{\gamma}$. By Lemma 101, there are constants Q and c such that the test path over any straight segment is (Q, c)-quasigeodesic. As each endpoint of a straight segment is within distance K_1 of $\overline{\gamma}$, we may extend the straight segment at each end by paths of length at most K_1, so that the endpoints of the extended straight segment lie on $\overline{\gamma}$, and furthermore, this new path is $(Q, c + K_1)$-quasigeodesic.

By stability for quasigeodesics, there is a constant L such that any $(Q, c+K_1)$-quasigeodesic is contained in an L-neighborhood of any geodesic connecting its endpoints. In particular, every straight segment is contained in an L-neighborhood of $\overline{\gamma}$. As straight segments are dense in γ, the result follows. □

6.1.2 The test path makes uniform progress

We now prove that the test path makes uniform progress, verifying the second condition in Theorem 97.

As the Cannon-Thurston pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$ is quasi-isometric to the hyperbolic metric, it suffices to show that τ_γ makes (L, M)-uniform progress along $\overline{\gamma}$, for any $L > 0$ and for $M = Q_M + c_M$, where Q_M and c_M are the quasi-isometry constants between $\widetilde{S}_h \times \mathbb{R}$ and $\mathbb{H}^3$. In fact, we will show:

Proposition 102. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then for any constant $M \geqslant 0$ there is a constant $L \geqslant 0$ such that for any non-exceptional geodesic γ, the test path τ_γ makes (L, M)-uniform progress along $\overline{\gamma}$.*

We now give a brief overview of the argument.

1. Suppose that the point $\gamma_\tau(t_1)$ is an (r, K)-bottleneck for $U_1 = F(\gamma((-\infty, a_1))$ and $V_1 = F(\gamma([b_1, \infty)))$, and similarly the point $\gamma_\tau(t_2)$ is an (r, K)-bottleneck for $U_2 = F(\gamma((-\infty, a_2))$ and $V_1 = F(\gamma([b_2, \infty)))$. Then both points $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ are bottlenecks for $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$, and furthermore, the distance between U and V is at least $d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(t_1), \tau_\gamma(t_2)) - 2K$. This gives lower bounds on the distances between points on the test path. It remains to show that for any two points on the test path sufficiently far apart along τ_γ, there are a pair of bottlenecks a definite distance apart in $\widetilde{S}_h \times \mathbb{R}$.

2. Let $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ be two points on the test path sufficiently far apart. Suppose the corresponding segment $\gamma([t_1, t_2])$ contains a long straight segment $\gamma(I)$. Then the test path $\tau_\gamma(I)$ is quasigeodesic, and so it has definite length. Furthermore, the corner segments at each end of $\gamma(I)$ give a pair of bottlenecks. Therefore, $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ are a definite distance apart in $\widetilde{S}_h \times \mathbb{R}$.

3. Now suppose that the corresponding segment $\gamma([t_1, t_2])$ of the geodesic in $\widetilde{S}_h$ does not contain any long straight segments. This implies that there is a constant L such that every segment of the test path over $\gamma([t_1, t_2])$ with length L contains a corner segment. By Lemma 100, it has a tame bottleneck. The bottleneck sets for successive tame bottlenecks are nested and so the distance in $\widetilde{S}_h \times \mathbb{R}$ increases linearly in the number of bottlenecks. Thus the test path makes definite progress, as required.

We start with Step 1 by showing that pairs of bottlenecks separate points along the test path τ_γ.

Proposition 103. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Let $\gamma_\tau(t_1)$ be an (r, K)-bottleneck for $U_1 = F(\gamma((-\infty, a_1])$ and $V_1 = F(\gamma([b_1, \infty)))$, and similarly let $\gamma_\tau(t_2)$ be an (r, K)-bottleneck for $U_2 = F(\gamma((-\infty, a_2])$ and $V_1 = F(\gamma([b_2, \infty)))$.*

Let $a = \min\{a_1, a_2\}$ and let $b = \max\{b_1, b_2\}$. Then for any two points p and q separated by $[a, b]$, the distance in $\widetilde{S}_h \times \mathbb{R}$ between $\tau_\gamma(p)$ and $\tau_\gamma(q)$ is at least the distance between $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$, up to bounded error $2K$, i.e.

$$d_{\widetilde{S}_n \times \mathbb{R}}(\tau_\gamma(p), \tau_\gamma(q)) \geqslant d_{\widetilde{S}_n \times \mathbb{R}}(\tau_\gamma(t_1), \tau_\gamma(t_2)) - 2K.$$

Proof. We may assume that $p \leqslant a \leqslant b \leqslant q$. Let η be a geodesic in $\widetilde{S}_h \times \mathbb{R}$ connecting $\tau_\gamma(p)$ and $\tau_\gamma(q)$.

By Lemma 100, there are constants r and K such that the points $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ are (r, K)-bottlenecks for $U = F(\gamma((-\infty, a_1]) \cap F(\gamma((-\infty, a_2]) = F(\gamma((\infty, a]))$ and $V = F(\gamma([b, \infty)) = F(\gamma([b_1, \infty)) \cap F(\gamma([b_2, \infty)))$.

As $\tau_\gamma(p) \in U$ and $\tau_\gamma(q) \in V$, the geodesic η passes within distance K of both $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$, so the length of η is at least $d_{\widetilde{S}_n \times \mathbb{R}}(\tau_\gamma(t_1), \tau_\gamma(t_2)) - 2K$, as required. □

By Lemma 98, the test path τ_γ is contained in a K-neighborhood of $\overline{\gamma}$. Hence, for an point p on the test path, its nearest point projection to $\overline{\gamma}$ is distance at most K away. Therefore, the distance between the nearest points on $\overline{\gamma}$ of any points p and q on the test path is at least $d_{\widetilde{S}_h \times \mathbb{R}}(p, q) - 2K$ apart, so it suffices to estimate distance between points on the test path in $\widetilde{S}_h \times \mathbb{R}$.

For Step 2, suppose that a long segment $\tau_\gamma(I)$ contains a long straight subsegment.

Proposition 104. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then for any constant M there is a constant L such that for any straight segment I and any interval $[p, q]$ such that $\tau_\gamma([p, q]) \cap \tau_\gamma(I)$ has arc length at least L, the distance in $\widetilde{S}_h \times \mathbb{R}$ between $\tau_\gamma(p)$ and $\tau_\gamma(p)$ is at least M.*

Proof. Let $I = [t_1, t_2]$. By Lemma 101, there are constants Q and c such that the test path $\tau_\gamma(I)$ over the straight segment is (Q, c)-quasigeodesic. By Definition 84, $\gamma(I)$ has corner segments at both ends. By Lemma 100, there are constants r and K, and points $u_i \leqslant t_i \leqslant v_i$ such that the endpoints $\tau_\gamma(t_i)$ are (r, K)-bottlenecks with respect to $U_i = F(\gamma((-\infty, u_i]))$ and $V_i = F(\gamma([v_i), \infty))$. Furthermore, the lengths of the segments $\tau_\gamma([u_i, t_i])$ and $\tau_\gamma([t_i, v_i])$ are at most K. In particular, the segment $\tau_\gamma([u_1, v_2])$, lying between the initial quadrant for t_1 and the terminal quadrant for t_2, is $(Q, c + 2K)$-quasigeodesic.

Suppose that $[p, q] \subseteq [u_1, v_2]$. Since $\tau_\gamma([u_1, v_2])$ is $(Q, c + 2K)$-quasigeodesic and the arc length of $\tau_\gamma([p, q]) \cap \tau_\gamma(I)$ is at least L, the distance in $\widetilde{S}_h \times \mathbb{R}$ between $\tau_\gamma(p)$ and $\tau_\gamma(q)$ is at least $L/Q - c - 2K$.

Now suppose that $[p, q]$ is not contained in $[u_1, v_2]$. Then at least one endpoint of $[p, q]$ lies outside $[u_1, v_2]$. Up to reversing the orientation on γ, we may assume that $p < u_1$, and hence $\tau_\gamma([p, q]) \cap \tau_\gamma(I) = \tau_\gamma([t_1, q])$. By assumption, the arc length of $\tau_\gamma([t_1, q])$ is at least L. So the distance in $\widetilde{S}_h \times \mathbb{R}$ between $\tau_\gamma(t_1)$ and $\tau_\gamma(q)$ is at least $L/Q - c - 2K$. The point $\tau_\gamma(p)$ lies in U_1, and so by the tame bottleneck property any geodesic from $\tau_\gamma(p)$ to $\tau_\gamma(q)$ passes within distance K of $\tau_\gamma(t_1)$. So the distance from $\tau_\gamma(p)$ to $\tau_\gamma(q)$ is at least $L/Q - c - 3K$.

The result follows if given M we choose $L = Q(M + c + 3K)$, where Q, c and K only depend on $(S_h\Lambda)$. □

We now assume that a segment $\tau_\gamma(I)$ contains no straight segment of arc length greater than L. By Definition 84, it follows that every segment $\tau_\gamma(I)$ of arc length L contains a corner segment. We now show that in this case the test path makes definite progress.

Proposition 105. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. For constants L and M as in Proposition 104 there is a constant N such that for any segment I of length NL in which every segment of $\tau_\gamma(I)$ of arc length L contains a corner segment, the distance in $\widetilde{S}_h \times \mathbb{R}$ between the endpoints of $\tau_\gamma(I)$ is at least M.*

Proof. Let $I_1, \ldots, I_n$ be consecutive intervals along I such that each test path segment $\tau_\gamma(I_j)$ has arc length L, and each segment $\gamma(I_j)$ contains a corner segment. By lemma 88, each corner segment is an (r, K)-bottleneck, with respect to a pair of sets $U_j = F((-\infty, u_j])$ and $V_j = F([v_j, \infty))$, with the arc length of $\tau_\gamma([u, v])$ at most K. We may assume that L is greater than K, and so for any pair of intervals I_j and I_{j+2}, the segments $\tau_\gamma(u_j, v_j)$ and $\tau_\gamma(u_{j+2}, v_{j+2})$ are disjoint, and the bottleneck sets are nested, $U_j \subset U_{j+2}$ and $V_{j+2} \subset V_j$. In particular, the distance between U_j and V_{j+2} is at least $2r$. The result now follows by choosing $N \geqslant M/r + 2$. ☐

We conclude the proof that test paths are quasigeodesics, assuming Lemma 100 and Lemma 101. The remainder of this section is devoted to the proof of these two results.

6.2 Properties of the height function

In this section, we specify the exact choice θ_Λ for the height function in Definition 54. To do so, we first define a function that is intermediate between the height function and the distance in $\mathrm{PSL}(2, \mathbb{R})$, which we call the radius function. We discuss its geometric interpretation, and use it to show that both the radius and the corresponding height functions are Lipschitz. We also show that both the radius and height functions at a point of intersection of a geodesic γ with a leaf of an invariant lamination can be estimated in terms of the angle of intersection.

6.2.1 The radius function

In Definition 54, the height function is defined in terms of the distance in $\mathrm{PSL}(2, \mathbb{R})$ from the geodesic to the extended laminations. Intuitively, we can think of distances in $\mathrm{PSL}(2, \mathbb{R})$ as extending the angle of intersection between the geodesic and the laminations to a continuous function along the geodesic. We define an intermediate function, which we call the radius function, to be roughly the exponential of the height function, or equivalently, the logarithm of the distance function in $\mathrm{PSL}(2, \mathbb{R})$. Intuitively, this extends the length of the projection interval to a continuous function along γ. We give below the precise definition, and then use it to estimate the radius function at an intersection point between a geodesic γ and a lamination in terms of the angle of intersection.

The angle θ of intersection between γ and a leaf $\ell \in \Lambda$ determines the radius of both the exponential interval E_ℓ and the projection interval I_ℓ. The radii of these intervals is equal to $\log \frac{1}{\theta}$, up to bounded additive error. For a single leaf ℓ, we define the *radius function* $\rho_{\gamma,\ell}$ to be

$$\rho_{\gamma,\ell}(t) = \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1)} \right\rfloor_1, \tag{13}$$

where ℓ^1 is the lift of ℓ in $\mathrm{PSL}(2, \mathbb{R})$. Up to bounded additive error, the value of the radius function at a point of intersection equals the radius of the projection interval for the leaf of intersection. At other points t, again up to bounded additive error, the value of the radius function equals the largest radius of an interval centered at t that is contained in the projection interval.

As the distance to one of the extended laminations is the infimum of the distance to any leaf of the laminations, we may define radius functions for the extended laminations as follows,

$$\rho_{\gamma,\overline{\Lambda}_+}(t) = \sup_{\ell \in \overline{\Lambda}_+} \rho_{\gamma,\ell}(t) \text{ and } \rho_{\gamma,\overline{\Lambda}_-}(t) = \sup_{\ell \in \overline{\Lambda}_-} \rho_{\gamma,\ell}(t).$$

We now estimate the radius function for a lamination at an intersection point using the radius function for the leaf of intersection. The exponential bounds on the distance between the lifts of two geodesics to the unit tangent bundle, from Proposition 69, become linear bounds for the logarithm of the reciprocal of the distance function. In particular, taking logarithms of (12) gives

$$\log \tfrac{1}{\theta} - |t| - \log L_0 \leqslant \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1)} \leqslant \log \tfrac{1}{\theta} - |t| + \log L_0, \tag{14}$$

and these bounds hold for $|t| \leqslant \log \frac{1}{\theta}$.

Suppose that in $\mathrm{PSL}(2,\mathbb{R})$ the point on γ closes to ℓ is $\gamma(t_\ell)$, with $d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma(t), \ell) = \theta_\ell$. Recall that the *exponential interval* E_ℓ for ℓ is $[t_\ell - \frac{1}{\log \theta_\ell}, t_\ell + \log \frac{1}{\theta_\ell}]$. For a compact interval $I \subset \mathbb{R}$ with length $|I|$ and midpoint m, define the *absolute value function* $|\cdot|_I$ to be $|t|_I = \lfloor |I| - |t| \rfloor_0$, as illustrated in Figure 9. We remark that for $t \in I$, $|t|_I$ is equal to the distance from t to the nearest endpoint of I, so for any $t \in I$, the interval $[t - |t|_I, t + |t|_I] \subseteq I$.

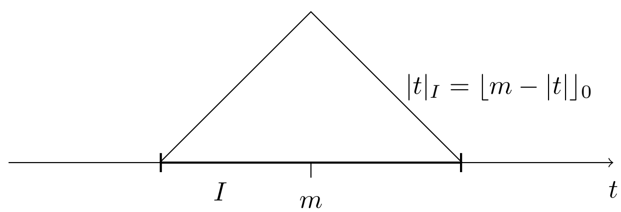

Figure 9: An absolute value function.

With this notation, we may rewrite (14) as

$$|t|_{E_\ell} - K \leqslant \rho_{\gamma,\ell}(t) \leqslant |t|_{E_\ell} + K, \tag{15}$$

where $K = \log L_0$.

We now use the above observations to show that if the value of the radius function $\rho_{\gamma,\ell}(t)$ is sufficiently large, then there is an interval centered at t of radius $\rho_{\gamma,\ell}(t)$ (up to bounded additive error) contained in the exponential interval $E_\ell \subset \gamma$ determined by the leaf ℓ.

Proposition 106. *There is a constant K such that for any geodesics γ and ℓ in $\mathbb{H}^2$, with unit speed parametrizations, then if $\rho_{\gamma,\ell}(t) \geqslant K$, then the interval centered at t of radius $\rho_{\gamma,\ell}(t) - K$ is contained in the exponential interval $E_\ell \subset \gamma$ determined by ℓ.*

Proof. We shall choose $K = \log L_0$, where L_0 is the constant from (12).

Suppose that in $\mathrm{PSL}(2,\mathbb{R})$ the closest point on γ to ℓ is $\gamma(t_\ell)$ and let the closest distance be θ_ℓ. Then the exponential interval is $E_\ell = [t_\ell - \log \frac{1}{\theta_\ell}, t_\ell + \log \frac{1}{\theta_\ell}]$. Using (15), if $\rho_{\gamma,\ell}(t) \geqslant K$, then $t \in E_\ell$ and $|\rho_{\gamma,\ell}(t) - |t|_{E_\ell}| \leqslant K$. For points $t \in E_\ell$, the value of $|t|_{E_\ell}$ is equal to the distance from t to the nearest endpoint of E_ℓ, and so

$$[t - |t|_{E_\ell}, t + |t|_{E_\ell}] \subseteq E_\ell.$$

Again, using (15) to estimate $\rho_{\gamma,\ell}(t)$ in terms of the absolute value function $|t|_{E_\ell}$ gives

$$[t - (\rho_{\gamma,\ell}(t) - K), t + (\rho_{\gamma,\ell}(t) - K)] \subseteq E_\ell,$$

as required. ∎

We now show that if $\gamma(t_1)$ is an intersection point for γ with a geodesic ℓ_1, and if the radius function at t_1 for another geodesic ℓ_2 is sufficiently larger than $\rho_{\gamma,\ell_1}(t_1)$, then the endpoints of ℓ_1 separate the endpoints of ℓ_2 and hence ℓ_1 and ℓ_2 intersect.

Proposition 107. *There is a constant $K \geqslant 1$, such that for any geodesics γ and ℓ_1 in $\mathbb{H}^2$ which intersect at $\gamma(t_1)$ at angle θ_1, then for any other geodesic ℓ_2, if the radius function at t_1 is sufficiently large, i.e. $\rho_{\gamma,\ell_2}(t_1) \geqslant \log \frac{1}{\theta_1} + K$, then ℓ_1 and ℓ_2 intersect.*

Proof. Set $K = T_0 + K_1 + 1$, where T_0 and K_1 are respective constants from Proposition 25 and Proposition 106.

By Proposition 25, the projection interval $I_{\ell_1} \subset \gamma$ for ℓ_1 is contained in $[t_1 - \log \frac{1}{\theta_1} - T_0, t_1 + \log \frac{1}{\theta_1} + T_0]$. As we have chosen $K \geqslant K_1 + T_0 + 1$, and we have assumed that $\rho_{\gamma,\ell_2}(t_1) \geqslant \rho_{\gamma,\ell_1}(t_1) + K$, Proposition 106 implies that the exponential interval E_{ℓ_2} contains the interval centered at t_1 of radius $\log \frac{1}{\theta_1} + K - K_1 \geqslant \log \frac{1}{\theta_1} + T_0 + 1$, so $I_{\ell_1} \subset E_{\ell_2}$, where the inclusion is strict.

As the exponential interval E_{ℓ_2} is contained in the projection interval I_{ℓ_2}, this implies that the projection interval $I_{\ell_1} \subset I_{\ell_2}$. As ℓ_1 intersects γ, Proposition 27 implies that ℓ_1 and ℓ_2 intersect, as required. ☐

For later use, we record the following estimate of the radius function of the extended lamination at an intersection point in terms of the radius function for the leaf of intersection.

Proposition 108. *There is a constant $K \geqslant 1$, such that for any closed hyperbolic surface S_h and lamination $\overline{\Lambda}$, if a geodesic γ intersects a leaf $\ell \in \Lambda$ at angle θ at $\gamma(t)$, then*

$$\rho_{\gamma,\ell}(t) \leqslant \rho_{\gamma,\overline{\Lambda}}(t) \leqslant \rho_{\gamma,\ell}(t) + K, \tag{16}$$

where $\overline{\Lambda}$ is the extended lamination corresponding to Λ.

Proof. The left hand inequality follows directly from $\rho_{\gamma,\overline{\Lambda}}$ being the supremum of $\rho_{\gamma,\ell}$ over all leaves $\ell \in \overline{\Lambda}$.

If ℓ is a leaf of a (non-extended) lamination, then it is disjoint from all leaves in the corresponding extended lamination. Proposition 107 shows that the value of the radius function determined by ℓ, at the intersection point of ℓ and γ, is at least the radius function at that point determined by any other leaf of the extended lamination, up additive error at most K, where K is the constant from Proposition 107. ☐

6.2.2 The choice of constant for the height function

The constant θ_Λ in the definition of the height function needs to be chosen to be sufficiently small, and it depends on the hyperbolic metric S_h and the pair of regular laminations Λ. We now give an explicit choice of θ_Λ which suffices for our purposes. For closed subsets A and B of the unit tangent bundle $T^1(S_h)$, let

$$d_{\mathrm{PSL}(2,\mathbb{R})}(A, B) = \min\{d_{\mathrm{PSL}(2,\mathbb{R})}(a, b) \mid a \in A, b \in B\}.$$

The constant θ_Λ needs to be less than half the distance between the extended laminations. However, our argument uses various properties of the geometry of the laminations, and so θ_Λ will also depend on:

1. The constant α_Λ from Proposition (11.1), giving the smallest angle of intersection between leaves of the two laminations.
2. The constant L_Λ from Proposition (11.3), giving the diameter of the compact complementary regions of $S_h \setminus (\Lambda_+ \cup \Lambda_-)$.
3. The constant T_0 from Proposition 25, giving the size of the nearest point projection intervals between geodesics in $\mathbb{H}^2$.

4. The constant ρ_Λ from Proposition 26, giving an upper bound on the diameter of the overlap between the nearest point projections of any two intersecting geodesics $\ell^+ \in \Lambda^-$ and $\ell^- \in \Lambda^-$ to any other geodesic γ.

5. The constants Q_Λ and c_Λ from Proposition 38 giving the quasi-isometry between the hyperbolic metric $d_{\mathbb{H}^2}$ and the Cannon-Thurston pseudometric $d_{\widetilde{S}_h}$ on $\widetilde{S}_h$.

6. The constants θ_0 and L_0 from Proposition 69, giving the bi-Lipschitz bounds on the rates of divergence of lifts of close geodesics.

7. The constant D_Λ from Proposition 79, giving an upper bound on the diameter of any innermost polygon.

8. The constant θ_P from Proposition 80, which ensures that the lift of a non-exceptional geodesic is close in $T^1(S_h)$ to at most one of the extended leaves in an innermost non-rectangular polygon.

All constants above depend only on (S_h, Λ), and do not depend on the non-exceptional geodesic γ. We now define θ_Λ.

Definition 109. Let

$$\theta_{\min} = \min\{\alpha_\Lambda, \rho_\Lambda, \tfrac{1}{2} d_{\mathrm{PSL}(2,\mathbb{R})}(\overline{\Lambda}^1_-, \overline{\Lambda}^1_+), \theta_0, \tfrac{1}{L_0}, \theta_P, 1\},$$

and then set

$$\theta_\Lambda = \theta_{\min}^6 e^{-6(T_0 + L_\Lambda + 3\rho_\Lambda + D_\Lambda + Q_\Lambda c_\Lambda)},$$

where all of these constants from Propositions (11.3), 25, 26, 38, 69, 79 and 80 only depend on (S_h, Λ).

6.2.3 Estimating the height function

If $\gamma(t)$ is a point of intersection of γ and a leaf ℓ of one of the (non-extended) laminations, then we can use the angle of intersection between γ and ℓ to estimate the value of the height function at the intersection point. In fact, we can define and use a (signed) height function for a single leaf.

We define the (signed) height function for a single leaf ℓ of one of the invariant laminations, as follows,

$$h_{\gamma,\ell}(t) = \begin{cases} \log_k \left\lfloor \rho_{\gamma,\ell}(t) - \log \frac{1}{\theta_\Lambda} \right\rfloor_1 & \text{if } \ell \in \Lambda_+ \\ -\log_k \left\lfloor \rho_{\gamma,\ell}(t) - \log \frac{1}{\theta_\Lambda} \right\rfloor_1 & \text{if } \ell \in \Lambda_- \end{cases},$$

where again, by our choice of θ_Λ, at most one of the terms on the right hand side above may be non-zero.

We can also rewrite the height function in terms of the radius functions for the extended laminations,

$$h_\gamma(t) = \log_k \left\lfloor \rho_{\gamma,\overline{\Lambda}_+}(t) - \log \tfrac{1}{\theta_\Lambda} \right\rfloor_1 - \log_k \left\lfloor \rho_{\gamma,\overline{\Lambda}_-}(t) - \log \tfrac{1}{\theta_\Lambda} \right\rfloor_1 .$$

We now show that at an intersection point, the signed height function for the leaf of intersection, which depends only on the angle of intersection, can be used to approximate the height function for the invariant laminations.

Proposition 110. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant K such that if γ is any non-exceptional geodesic γ in $\widetilde{S}_h$, and ℓ^+ is a leaf of the invariant lamination Λ_+ which intersects γ at $\gamma(t)$ at angle θ, then*

$$\begin{aligned} h_{\gamma,\ell^+}(t) \leqslant h_\gamma(t) &\leqslant h_{\gamma,\ell^+}(t) + K & \text{if } \theta \leqslant \theta_\Lambda \\ h_\gamma(t) &\leqslant K & \text{if } \theta \geqslant \theta_\Lambda. \end{aligned}$$

Similarly, if $\gamma(t)$ is an intersection point of γ with a leaf $\ell^- \in \Lambda_-$ of angle θ, then

$$\begin{aligned} h_{\gamma,\ell^-}(t) - K \leqslant h_\gamma(t) &\leqslant h_{\gamma,\ell^-}(t) && \text{if } \theta \leqslant \theta_\Lambda \\ -K &\leqslant h_\gamma(t) && \text{if } \theta \geqslant \theta_\Lambda. \end{aligned}$$

We remark that in Proposition 110, although the definition of the height function depends on the cutoff constant θ_Λ, the additive error constant K depends only on (S_h, Λ).

As the radius functions determine the height function, we can now complete the proof of Proposition 110, showing that the value of the height function at an intersection point is equal to the value of the leafwise height function for the leaf of intersection at the intersection point, up to bounded additive error.

Proof (of Proposition 110). Up to reparametrizing γ by a translation, suppose that $\gamma(0)$ is an intersection point of γ with $\ell_1 \in \Lambda_+$ with angle θ. The argument is the same in the other case up to swapping the laminations and reversing the sign of the height function.

We choose $K = \log_k K_1$, where $K_1 \geqslant 1$ is the constant from Proposition 107.

We first show the upper bound. As $\overline{\Lambda}_+$ is closed, there is a leaf ℓ_2 in the extended lamination $\overline{\Lambda}_+$, realizing the height function, i.e. $h_\gamma(0) = h_{\gamma,\ell_2}(0)$. As ℓ_1 is a leaf of Λ_+, is it disjoint from all other leaves in the extended lamination $\overline{\Lambda}_+$. Therefore, by Proposition 107 the radius function for ℓ_1 is a coarse upper bound for the radius function for ℓ_2 at $t = 0$, i.e.

$$\rho_{\gamma,\ell_2}(0) \leqslant \rho_{\gamma,\ell_1}(0) + K_1,$$

where K_1 is the constant from Proposition 107. Subtracting $\log \frac{1}{\theta_\Lambda}$ from each side gives

$$\rho_{\gamma,\ell_2}(0) - \log \tfrac{1}{\theta_\Lambda} \leqslant \rho_{\gamma,\ell_1}(0) - \log \tfrac{1}{\theta_\Lambda} + K_1.$$

Using the elementary observation that $\lfloor x + y \rfloor_1 \leqslant \lfloor x \rfloor_1 + \lfloor y \rfloor_1$, and as $K_1 \geqslant 1$,

$$\left\lfloor \rho_{\gamma,\ell_2}(0) - \log \tfrac{1}{\theta_\Lambda} \right\rfloor_1 \leqslant \left\lfloor \rho_{\gamma,\ell_1}(0) - \log \tfrac{1}{\theta_\Lambda} \right\rfloor_1 + K_1.$$

As $\log_k(x)$ is $(1/\log k)$-Lipschitz for $x \geqslant 1$,

$$h_{\gamma,\ell_2}(0) \leqslant h_{\gamma,\ell_1}(0) + K_1/\log k,$$

as required.

For the lower bound, if $\theta \leqslant \theta_\Lambda$, then by Definition 54, $h_{\gamma,\ell_1^+}(t)$ is a lower bound for $h_\gamma(t)$. However, if $\theta \geqslant \theta_\Lambda$, then the contribution of distance to leaves of Λ_+ to the height function may be zero, and the height function may be determined by distance to leaves of $\overline{\Lambda}_-$, and so then there is no lower bound. ☐

6.2.4 The radius function is Lipschitz

A function $f \colon \mathbb{R} \to \mathbb{R}$ is c-*Lipschitz* if $|f(x) - f(y)| \leqslant c|x - y|$, and for differentiable functions this is equivalent to $|f'(x)| \leqslant c$. In this section, we show that the radius function is 1-Lipschitz.

Proposition 111. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Suppose that γ is a non-exceptional geodesic in $\widetilde{S}_h$ with unit speed parametrization $\gamma(t)$. Then the radius functions $\rho_\gamma(t), \rho_{\gamma,\Lambda_+}(t)$ and $\rho_{\gamma,\Lambda_-}(t)$ are all 1-Lipschitz.*

In fact, as the derivative of $\log_k(x)$ takes values between 0 and $1/\log k$ for $x \geqslant 1$, Proposition 111 also implies that the height function, which is defined in terms of log of the radius function, is $(1/\log k)$-Lipschitz, which we record for future reference.

Corollary 112. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. The for any non-exceptional geodesic γ with unit speed parametrization $\gamma(t)$, the height function $h_\gamma(t)$ is $(1/\log k)$-Lipschitz.* □

Proof. Recall the definition of the height function,

$$h_\gamma(t) = \log_k \left\lfloor \rho_{\gamma,\overline{\Lambda}_+}(t) - \log \tfrac{1}{\theta_\Lambda} \right\rfloor_1 - \log_k \left\lfloor \rho_{\gamma,\overline{\Lambda}_-}(t) - \log \tfrac{1}{\theta_\Lambda} \right\rfloor_1 .$$

If $f(x)$ is 1-Lipschitz, then $\lfloor f(x) - a \rfloor_b$ is also 1-Lipschitz for any a and b. As the derivative of $\log_k(x)$ takes values in $(0, 1/\log k]$ for $x \geqslant 1$, each term on the right hand side above is $(1/\log k)$-Lipschitz. As a sum or difference of $(1/\log k)$-Lipschitz functions is $(1/\log k)$-Lipschitz, the result follows. □

Suppose that γ is a geodesic with unit speed parametrization $\gamma\colon \mathbb{R} \to \widetilde{S}_h$. Suppose that R is either an ideal complementary region of one of the invariant laminations, or a compact complementary region of their union. We define the *intersection interval* I_R to be the closure of the pre-image $\gamma^{-1}(R)$. If R is an innermost polygon, then we say that I_R is an *innermost intersection interval*, i.e. the interior of $\gamma(I_R)$ is disjoint from the invariant laminations.

From Corollary 112 we deduce below that the test path over an innermost intersection interval has bounded arc length.

Corollary 113. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant K, such that for any innermost intersection interval $\gamma(I_R)$, the arc length of $\tau_\gamma(I_R)$ is at most K.*

Proof. By Proposition (11.3), as the interior of $\gamma(I_R)$ is disjoint from both laminations, the hyperbolic length of $\gamma(I_R)$ is at most L_Λ. By Corollary 112 the height function is $(1/\log k)$-Lipschitz. As $\gamma(I_R)$ is disjoint from the invariant laminations, its length is determined by the vertical z-coordinate, so the length of $\tau_\gamma(I_R)$ is at most $K = L_\Lambda / \log k$, which only depends on (S_h, Λ), as required. □

We prove below that the radius function determined by a single leaf is 1-Lipschitz. Since the height is defined in terms of distance in the unit tangent bundle to the two extended invariant laminations, and the distance to an extended lamination is the infimum of the distance to all of the leaves in the lamination, the required Lipschitz property for the the height function will follow from the fact that the supremum of 1-Lipschitz functions is 1-Lipschitz.

Proposition 114. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then for any geodesic γ with unit speed parametrization $\gamma(t)$, and any distinct geodesic ℓ, the radius function $\rho_{\gamma,\ell}(t)$ is 1-Lipschitz.*

Proof. We simplify notation by setting $d(t) = d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1)$. It suffices to bound the derivative of $\log d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1) = \log d(t)$, as this is equal to the negative of the radius function where $d(t) \leqslant 1/e$. When $d(t) \geqslant 1/e$, the radius function is the constant function 1, and is automatically 1-Lipschitz.

Let α be a geodesic arc in $\mathrm{PSL}(2,\mathbb{R})$, realizing $d(t) = d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1)$, i.e. $\mathrm{length}(\alpha) = d(t)$. Let $\phi_t\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ be the geodesic flow on $\mathrm{PSL}(2,\mathbb{R})$. Then $\phi_h\alpha$ is a path in $\mathrm{PSL}(2,\mathbb{R})$ from $\gamma(t+h)$ to ℓ. It is well known that the geodesic flow ϕ_h in $\mathrm{PSL}(2,\mathbb{R})$ expands or contracts distances by at most e^h, see for example [Man91, page 75]. So $\mathrm{length}(\phi_h\alpha) \leqslant e^h d(t)$. Since the length of $\phi_t\alpha$ is an upper bound for the distance from $\gamma(t+h)$ to ℓ, we have

$$d(t+h) \leqslant e^h d(t). \tag{17}$$

Similarly, let β be a geodesic in $\mathrm{PSL}(2,\mathbb{R})$ realizing the distance from $\gamma(t+h)$ to ℓ. Then $\phi_{-h}\beta$ is a path from $\gamma(t)$ to ℓ. This gives an upper bound of $e^h\mathrm{length}(\beta)$ on the distance from $\gamma(t)$ to ℓ, and hence

$$d(t) \leqslant e^h d(t+h). \tag{18}$$

Combining (17) and (18) gives

$$\log e^{-h} d(t) - \log d(t) \leqslant \log d(t+h) - \log d(t) \leqslant \log e^{h} d(t) - \log d(t)$$

which simplifies to

$$|\log d(t+h) - \log d(t)| \leqslant |h|.$$

Thus, the radius function determined by a single geodesic is 1-Lipschitz, as required. □

We may now complete the proof of Proposition 111.

Proof of Proposition 111. As $\rho_\gamma(t) = \max\{\rho_{\gamma,\Lambda_+}(t), \rho_{\gamma,\Lambda_-}(t)\}$ it suffices to show that $\rho_{\gamma,\Lambda_+}(t)$ and $\rho_{\gamma,\Lambda_-}(t)$ are 1-Lipschitz. We give the argument for Λ_+, the same argument works for Λ_-.

The distance from $\gamma^1(t)$ to the extended lamination $\overline{\Lambda}^1_+$ is the infimum of distances to each leaf $\ell^1 \in \overline{\Lambda}^1_+$, that is

$$d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \overline{\Lambda}^1_+) = \inf_{\ell \in \overline{\Lambda}_+} d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1).$$

Recall the definition of the radius function,

$$\rho_{\gamma,\Lambda_+}(t) = \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \overline{\Lambda}^1_+)} \right\rfloor_1,$$

as the reciprocal function is decreasing, and the logarithm function is increasing,

$$\rho_{\gamma,\Lambda_+}(t) = \sup_{\ell \in \overline{\Lambda}_+} \rho_{\gamma,\ell}(t).$$

The radius function for an individual leaf is 1-Lipschitz by Proposition 114, and a supremum of 1-Lipschitz functions is 1-Lipschitz, and so the radius function is 1-Lipschitz, as required. □

6.3 Tame bottlenecks

In this section, we prove Lemma 100 that corner segments create tame bottlenecks. To do so, we first define transverse rectangles for a geodesic, namely rectangles of positive measure such that the geodesic crosses all leaves containing the sides of the rectangle. We then show that transverse rectangles give rise to bottlenecks. Our construction does not come with a bound on the arc length of the test path segment between the bottleneck sets. However, we show that by making the rectangles smaller and thus increasing the size of the bottleneck sets, there is a transverse rectangle with a bound on arc length of the test path segment, i.e. the bottlenecks are tame.

6.3.1 Transverse rectangles

Definition 115. Given a geodesic γ, we say that a rectangle of positive measure is a *transverse rectangle* for γ if γ crosses all leaves containing the sides of the rectangle.

A choice of unit speed parametrization for a geodesic γ orders the leaves of the laminations intersecting γ. Using the ordering and our conventions for rectangles illustrated in Figure 10, we specify some notation for transverse rectangles. Note that the geodesic need not itself intersect the rectangle.

Definition 116. Suppose that γ is a non-exceptional geodesic parametrized with unit speed. Suppose that γ intersects leaves ℓ_1 and ℓ_2 in either invariant lamination $\Lambda_+ \cup \Lambda_-$ at points $\gamma(t_1)$ and $\gamma(t_2)$, respectively. If $t_1 \leqslant t_2$ then we say that $\ell_1 \leqslant_\gamma \ell_2$.

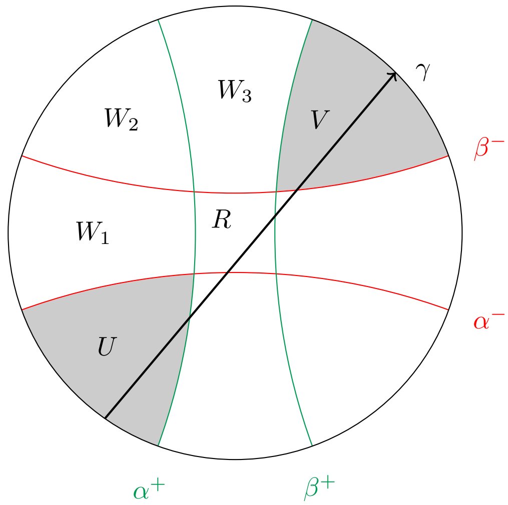

Figure 10: A transverse rectangle.

Suppose that R is a rectangle with positive measure transverse to a geodesic γ. The rectangle has two sides contained in leaves of Λ_+, which we shall label α^+ and β^+ so that $\alpha^+ \leqslant_\gamma \beta^+$. Similarly, the rectangle has two sides contained in leaves of Λ_-, which we shall label α^- and β^- so that $\alpha^- \leqslant_\gamma \beta^-$.

As γ is oriented, it has an initial limit point $\overline{\gamma}_-$ and a terminal limit point $\overline{\gamma}_+$. We call the quadrant whose limit set contains $\overline{\gamma}_-$ the *initial quadrant*, and the quadrant whose limit set contains $\overline{\gamma}_+$ the *terminal quadrant*. Using our notation illustrated in Figure 10, the region U, with boundary contained in α^+ and α^-, is the initial quadrant, and the region V, with boundary contained in β^+ and β^-, is the terminal quadrant.

Suppose that a non-exceptional geodesic γ is transverse to a rectangle R with optimal height z. The main result of this section is that the optimal height rectangle $F_z(R)$ is a (r, K)-*bottleneck* with respect to the flow sets over the initial and terminal quadrants. The constant K depends on the measure of the rectangle (as well as various constants depending on (S, Λ)), and tends to infinity as the area of the rectangle tends to zero. In particular, the geodesic $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$ with the same limit points as the path $\iota(\gamma)$ passes within a bounded distance of the square $F_z(R)$.

Lemma 117. *(Transverse rectangles create bottlenecks.) Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Suppose that γ is a non-exceptional geodesic that intersects a rectangle R with measure at least $A > 0$ and optimal height z. Then there are constants $r > 0$ and $K \geqslant 0$ (that depends on Λ and A) such that the optimal height rectangle $F_z(R) = R \times \{z\}$ is an (r, K)-bottleneck for the flow sets $F(U)$ and $F(V)$ over the initial and terminal quadrants of R.*

In particular, the geodesic $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$ with the same limit points as $\iota(\gamma)$ passes within Cannon-Thurston distance K of the optimal height rectangle $F_z(R)$ in $\widetilde{S}_h \times \mathbb{R}$.

6.3.2 Outermost rectangles for small angles

Suppose that a leaf ℓ intersects γ at $\gamma(t)$. Since the laminations are closed, there is a unique transverse rectangle containing $\gamma(t)$ whose side along ℓ has the largest measure among all such rectangles. We call this the *outermost* transverse rectangle determined by γ and ℓ. We show that as long as the angle between ℓ and γ is sufficiently small, the area of the outermost rectangle is bounded below.

Definition 118. Suppose that γ is a non-exceptional geodesic intersecting a leaf $\ell_- \in \Lambda_-$ at a point p. We say that a transverse rectangle R containing p is *outermost* if it has the following properties.

- The sides $\alpha_+ \leqslant_\gamma \beta_+$ of R are contained in the outermost leaves of Λ_+ intersecting both γ and ℓ_-.
- The sides $\alpha_- \leqslant_\gamma \beta_-$ are contained in the outermost leaves of Λ_- intersecting all α_+, β_+ and γ.

Similarly, if p is an intersection point of γ and a leaf ℓ_+ of Λ_+, we say a transverse rectangle R is *outermost* if it has the properties above with the two invariant laminations swapped.

Proposition 119. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Let Q_Λ and c_Λ be the constants from Proposition 38 and let θ_Λ be the constant from Definition 109. Then there are positive constants $A > 0$ and $K \geqslant 0$ such that for any non-exceptional geodesic γ in S_h, with unit speed parametrization, and with $\gamma(t)$ a point of intersection between γ and a leaf $\ell \in \Lambda_+$ with angle $\theta \leqslant \theta_\Lambda$, then the outermost transverse rectangle R determined by $\gamma \cap \ell$ has the following properties.*

(119.1) Let ℓ have a unit speed parametrization in the hyperbolic metric. Then the segment ℓ lying between α_- and β_- is an interval $\ell([-r_1, r_2])$ where $r_i = \log \frac{1}{\theta}$ up to additive error at most $\frac{1}{6} \log \frac{1}{\theta_\Lambda} - c_\Lambda$, where ℓ has unit speed parametrization with intersection point $\ell(0)$ with γ.

(119.2) The sides of R in Λ_+ have measure $dy(R)$ satisfying

$$0 < \tfrac{5}{3Q_\Lambda} \log \tfrac{1}{\theta_\Lambda} \leqslant \tfrac{2}{Q_\Lambda}(\log \tfrac{1}{\theta} - \tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda}) \leqslant dy(R) \leqslant 2Q_\Lambda(\log \tfrac{1}{\theta} + \tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda}).$$

(119.3) The measure of the rectangle R is at least $A = A_\Lambda/(3Q_\Lambda^2) > 0$.

(119.4) The sides of R in Λ_- have measure $dx(R)$ satisfying

$$0 < \frac{A_f}{2Q_\Lambda(\log \frac{1}{\theta} + \frac{1}{6} \log \frac{1}{\theta_\Lambda})} \leqslant dx(R) \leqslant \frac{A_\Lambda}{\frac{2}{Q_\Lambda}(\log \frac{1}{\theta} - \frac{1}{6} \log \frac{1}{\theta_\Lambda})},$$

where $A_\Lambda > 0$ is the constant from Proposition 81.

(119.5) The rectangle R has optimal height $\log_k \log \frac{1}{\theta}$ up to additive error at most K.

The same holds for γ intersecting a leaf ℓ of Λ_-, except bounds on the measures of the sides in Λ_+ and Λ_- are swapped, and the optimal height of the outermost transverse rectangle is $-\log_k \log \frac{1}{\theta}$ up to additive error at most K.

Proof. Given a hyperbolic metric S_h and a suited pair of laminations Λ, we recall various constants from previous results. Let $\theta_\Lambda > 0$ be the constant from Definition 109. In this argument, we will use the fact that $\theta_\Lambda \leqslant \alpha_\Lambda^2 e^{-2T_0 - 2L_\Lambda - 2Q_\Lambda c_\Lambda}$, where α_Λ is defined in Proposition (11.1), T_0 is the constant from Proposition 25, L_Λ is the constant from Proposition (11.3), and Q_Λ and c_Λ are the quasi-isometry constants from Proposition 38.

To simplify expressions, we will define a sequence of constants T_i during the argument. They will all depend only on $\mathrm{PSL}(2, \mathbb{R})$ or (S_h, Λ) and in particular, not on θ.

Suppose that a leaf ℓ of Λ_- intersects a non-exceptional geodesic γ at the point $\gamma(t)$ with angle $\theta \leqslant \theta_\Lambda$. We parametrize ℓ with unit speed so that the intersection point is $\ell(0)$.

By Proposition 25, the image of γ under nearest point projection to ℓ is equal to an interval $\ell(I_\gamma)$, where $I_\gamma = [-T_\gamma, T_\gamma]$ is a symmetric interval about $t = 0$, where T_γ satisfies $\log \frac{1}{\theta} \leqslant T_\gamma \leqslant \log \frac{1}{\theta} + T_0$.

By Proposition (11.3), there is a constant $L_\Lambda > 0$ such that every segment of ℓ of length L_Λ intersects a leaf of Λ_-. In particular, for any interval of ℓ with length L_Λ, nested sufficiently far inside the projection interval for γ onto ℓ,

- there will be a leaf of Λ_- intersecting the interval, and

- the nearest point projection interval of this leaf will be contained in the nearest point projection interval for γ on ℓ.

It follows that the endpoints of this leaf and the endpoints of γ are linked on the boundary circle, and so γ and the leaf intersect, as in Proposition 27.

Similarly, for any interval of ℓ of length L_Λ, sufficiently far from the projection interval for γ, the leaves that intersect the interval will have projection intervals onto ℓ which are disjoint from the projection interval of γ onto ℓ, and so these leaves are disjoint from γ.

We wish to produce leaves of Λ_- intersecting ℓ close to the endpoints of $\ell(I_\gamma)$. To do so, we begin by picking four intervals $I_1, \ldots, I_4$ in ℓ, all of length L_Λ, chosen so that

- the intervals $\ell(I_1)$ and $\ell(I_4)$ are the innermost among intervals in $\ell - \ell(I_\gamma)$ such that any leaf of Λ_+ intersecting $\ell(I_1)$ or $\ell(I_4)$ does not intersect γ; and
- the intervals $\ell(I_2)$ and $\ell(I_3)$ are the outermost among intervals in $\ell(I_\gamma)$ so that any leaf of Λ_+ that intersects them also intersects γ.

We claim that the choice of the intervals is possible by sufficient nesting that does not depend on θ. We choose T_1 to be an upper bound on the radius of the nearest projection of any leaf $\ell_+ \in \Lambda_+$ to any leaf $\ell_- \in \Lambda_+$. As the angle of intersections of leaves in Λ_+ with leaves in Λ_- is at most α_Λ by Proposition (11.1), we may choose

$$T_1 = T_0 + \log \tfrac{1}{\alpha_\Lambda}, \tag{19}$$

by Proposition 24. This choice of T_1 does not depend on the angle θ between γ and ℓ. We will choose the outer intervals I_1 and I_4 to be distance T_1 outside I_γ, and we will choose the inner intervals I_2 and I_3 to be nested distance T_1 inside I_γ. This is illustrated in Figure 11.

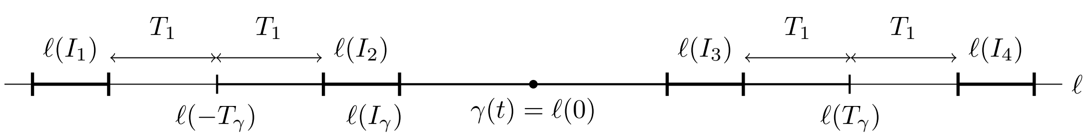

Figure 11: Projection intervals on ℓ.

In order for our construction to work, the projection interval I_γ for γ must be sufficiently large. We will require $T_\gamma \geqslant T_1 + L_\Lambda$. Equivalently, $T_\gamma = \log \frac{1}{\theta} \geqslant T_0 + \log \frac{1}{\alpha_\Lambda} + L_\Lambda$, which is satisfied as long as $\theta \leqslant \alpha_\Lambda e^{-T_0 - L_\Lambda}$. This is guaranteed by our choice of θ_Λ from Definition 109, and so T_γ is large enough to construct the nested intervals.

Consider now the first interval $\ell(I_1)$. Since its length is L_f, there is a leaf ℓ_1^- of Λ_- which intersects ℓ, say at $\ell(t_1)$ for t_1 in I_1. Since the angle of intersection between any two leaves is at least α_Λ, the nearest point projection interval $\ell(I_{\ell_1^-})$ onto ℓ is contained in a symmetric interval of radius $\log \frac{1}{\alpha_\Lambda} + T_0$ centered at t_1. This radius is equal to our choice of T_1, and so $\ell(I_{\ell_1^-})$ is disjoint from the projection interval $\ell(I_\gamma)$ for γ. In particular, ℓ_1^- is disjoint from γ. This is illustrated in Figure 12. Exactly the same argument shows that there is a leaf ℓ_4^- of Λ_- intersecting $\ell(I_4)$ which is disjoint from γ. In Figure 12 we have drawn $\gamma(t)$ in the interior of the rectangle we construct, but $\gamma(t)$ may in fact lie on the boundary, as ℓ may be a boundary leaf of the rectangle.

Now consider the second interval $\ell(I_2)$. As this interval has length L_f, there is a leaf ℓ_2^- of Λ_- which intersects ℓ, say at $\ell(t_2)$ in this interval $\ell(I_2)$. Again, as the angle of intersections is at least α_Λ, the nearest point projection interval $\ell(I_{\ell_2^-})$ for this leaf onto ℓ is contained in a symmetric interval of radius $\log \frac{1}{\alpha_\Lambda} + T_0$

centered at t_2. This radius is equal to our choice of T_1, and so $\ell(I_{\ell_2^-})$ is contained inside the projection interval $\ell(I_\gamma)$ for γ. Hence ℓ_2^- and γ intersect. Exactly the same argument shows that there is a leaf ℓ_3^- of Λ_- intersecting $\ell(I_3)$ which also intersects γ.

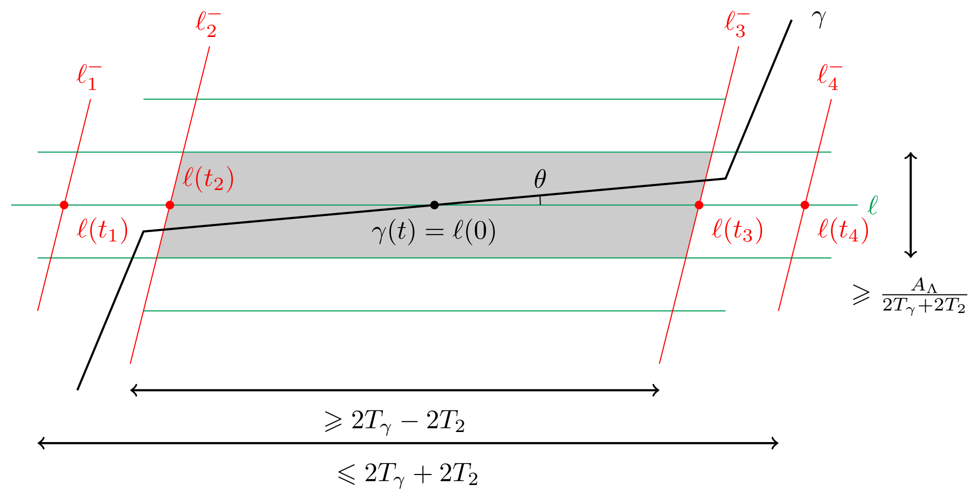

Figure 12: Small angles give long transverse rectangles.

The subinterval of ℓ between ℓ_2^- and ℓ_3^- will be contained in the rectangle we construct. We now verify that this subinterval satisfies the bounds on hyperbolic length from Proposition (119.1). Let ρ_θ be the minimum distance from $\ell(0)$ to either of the leaves ℓ_2^- and ℓ_3^-. By our choice of intervals above,

$$\log \tfrac{1}{\theta} - T_1 - L_\Lambda \leqslant \rho_\theta \leqslant \log \tfrac{1}{\theta} + T_0 - T_1.$$

Recall that $T_1 = T_0 + \log \frac{1}{\alpha_\Lambda}$, which gives

$$\log \tfrac{1}{\theta} - T_0 - \log \tfrac{1}{\alpha_\Lambda} - L_\Lambda \leqslant \rho_\theta \leqslant \log \tfrac{1}{\theta} - \log \tfrac{1}{\alpha_\Lambda}.$$

In order to show the bounds from Proposition (119.1), it therefore suffices to show that

$$\tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda} - c_\Lambda \geqslant T_0 + \log \tfrac{1}{\alpha_\Lambda} + L_\Lambda.$$

Equivalently,

$$\log \tfrac{1}{\theta_\Lambda} \geqslant 6(T_0 + \log \tfrac{1}{\alpha_\Lambda} + L_\Lambda + c_\Lambda). \tag{20}$$

and (20) follows directly from our choice of θ_Λ in Definition 109, which in fact satisfies the stronger estimate

$$\log \tfrac{1}{\theta_\Lambda} \geqslant 6(T_0 + \log \tfrac{1}{\alpha_\Lambda} + L_\Lambda + Q_\Lambda c_\Lambda), \tag{21}$$

as $Q_\Lambda \geqslant 1$.

The bound on the hyperbolic length of the subinterval of ℓ between ℓ_2^- and ℓ_3^- immediately gives the bound on the measure of the subinterval from Proposition (119.2), using the quasi-isometry between the hyperbolic and flat metric, Proposition 38. In particular, as $\theta \leqslant \theta_\Lambda$, the measure of this subinterval of ℓ is at least $\frac{2}{Q_\Lambda}(\log\frac{1}{\theta} - \frac{1}{6}\log\frac{1}{\theta_\Lambda}) \geqslant \frac{5}{3Q_\Lambda}\log\frac{1}{\theta_\Lambda}$, which is positive. Hence, by Proposition 81, the leaves ℓ_2^- and ℓ_3^- bound a maximal rectangle of area at least A_Λ containing this subinterval of ℓ.

By our choice of intervals, the distance between t_2 and t_3 is at least $2(T_\gamma - T_1 - L_\Lambda)$. Similarly, the distance between t_1 and t_4 is at most $2(T_\gamma + T_1 + L_\Lambda)$. For notational convenience, set

$$T_2 = T_1 + L_\Lambda. \tag{22}$$

Since the arc of ℓ between the intersection points with ℓ_1^- and ℓ_4^- contains the arc of ℓ between ℓ_2^- and ℓ_3^-, by Proposition 81, the leaves ℓ_1^- and ℓ_4^- also bound a rectangle of area at least A_Λ. Note that the hyperbolic length of the arc of ℓ between ℓ_1^- and ℓ_4^- is at most $2(T_\gamma + T_2)$. So the horizontal measure of the rectangle bounded by ℓ_1^- and ℓ_4^- is at most $2Q_\Lambda(T_\gamma + T_2) + c_\Lambda$. For notational convenience, set $T_3 = T_2 + \frac{1}{2}Q_\Lambda c_\Lambda$, and observe that (21) shows that

$$\log\tfrac{1}{\theta_\Lambda} \geqslant 6T_3 \geqslant 2T_3, \tag{23}$$

where we use the weaker lower bound to simplify constants in the following calculations. In particular, $T_\gamma - T_3 \geqslant \log\frac{1}{\theta_\Lambda} - T_3 \geqslant T_3 > 0$ is positive.

The vertical measure of the rectangle bounded by ℓ_1^- and ℓ_4^- is therefore at least $A_\Lambda/(2Q_\Lambda(T_\gamma+T_2)+c_\Lambda) \geqslant A_\Lambda/(2Q_\Lambda(T_\gamma + T_3))$, where the last inequality follows by our choice of T_3, and the fact that $Q_\Lambda \geqslant 1$.

The intersection of the two rectangles above is a transverse rectangle for γ. This is because, by construction, the leaves ℓ_2^- and ℓ_3^- intersect γ. Furthermore, as ℓ_1^- and ℓ_4^- are disjoint from γ, and lie on opposite sides of γ, every leaf of Λ_+ which intersects both ℓ_1^- and ℓ_4^- also intersects γ. The measure of the arc of ℓ between ℓ_2^- and ℓ_3^- is therefore equal to the measure of each side of the rectangle in Λ_+, and so Proposition (119.2) holds.

By the measure estimates above, the measure of the transverse rectangle is at least

$$A_\Lambda \frac{\frac{2}{Q_\Lambda}(T_\gamma - T_3)}{2Q_\Lambda(T_\gamma + T_3)} = \frac{A_\Lambda}{Q_\Lambda^2}\,\frac{T_\gamma - T_3}{T_\gamma + T_3}.$$

Since $T_\gamma \geqslant 2T_3$, by (23), it follows that the measure of the transverse rectangle is at least $\frac{1}{3Q_\Lambda^2}A_\Lambda$, so we may choose $A_1 = \frac{1}{3Q_\Lambda^2}A_\Lambda > 0$. This completes the proof of Proposition (119.3). Proposition (119.4) is an immediate consequence of Proposition (119.1) and Proposition (119.3).

We now estimate the optimal height of the rectangle to show the final statement Proposition (119.5). Recall that the optimal height z is $\frac{1}{2}\log_k(y/x)$, where x and y are the measures of the horizontal and vertical sides. Therefore

$$\tfrac{1}{2}\log_k \frac{\frac{2}{Q_\Lambda}(T_\gamma - T_3)}{\frac{A_\Lambda}{\frac{2}{Q_\Lambda}(T_\gamma - T_3)}} \leqslant z \leqslant \tfrac{1}{2}\log_k \frac{2Q_\Lambda(T_\gamma + T_3)}{\frac{A_\Lambda}{2Q_\Lambda(T_\gamma + T_3)}}, \tag{24}$$

which we may rewrite as

$$\log_k \tfrac{2}{Q_\Lambda\sqrt{A_\Lambda}}(T_\gamma - T_3) \leqslant z \leqslant \log_k 2\tfrac{Q_\Lambda}{\sqrt{A_\Lambda}}(T_\gamma + T_3). \tag{25}$$

We will use the following elementary bounds: for $x \geqslant a$, $x/2 \leqslant x - a$; and for $a \geqslant 2, b \geqslant 2$, $\log_k(a+b) \leqslant \log_k a + \log_k b$. This gives

$$\log_k T_\gamma - \log_k Q_\Lambda\sqrt{A_\Lambda} \leqslant z \leqslant \log_k T_\gamma + \log_k 2\tfrac{Q_\Lambda}{\sqrt{A_\Lambda}} + \log_k T_3$$

Using $\log \frac{1}{\theta} \leqslant T_\gamma \leqslant \log \frac{1}{\theta} + T_0$ gives

$$\log_k \log \tfrac{1}{\theta} - \log_k Q_\Lambda \sqrt{A_\Lambda} \leqslant z \leqslant \log_k \log \tfrac{1}{\theta} + \log_k T_0 + \log_k 2\tfrac{Q_\Lambda}{\sqrt{A_\Lambda}} + \log_k T_3. \tag{26}$$

We may choose $K = \max\{\log_k Q_\Lambda \sqrt{A_\Lambda}, \log_k T_0 + \log_k 2\frac{Q_\Lambda}{\sqrt{A_\Lambda}} + \log_k T_3\}$, as required.

Finally, if γ intersects a leaf of Λ_- instead of Λ_+, the bounds on the lengths of the horizontal and vertical measures of the rectangle are swapped but otherwise the entire argument above goes through. The bounds on the optimal height from (24) then become

$$\tfrac{1}{2} \log_k \frac{2Q_\Lambda(T_\gamma + T_3)}{\frac{A_\Lambda}{2Q_\Lambda(T_\gamma+T_3)}} \leqslant z \leqslant \tfrac{1}{2} \log_k \frac{\frac{A_\Lambda}{\frac{2}{Q_\Lambda}(T_\gamma - T_3)}}{\frac{2}{Q_\Lambda}(T_\gamma - T_3)}.$$

Multiplying the line above by -1 reverses the inequalities and takes the reciprocals of the fractions inside the log which exactly gives (25), except with the z replaced by $-z$. In particular, the bounds from (26) hold for $-z$, as required. □

6.3.3 Truncated rectangles for small angles

Suppose that $\gamma(t)$ is an intersection point with a sufficiently small angle of a non-exceptional geodesic with a leaf of an invariant lamination. By Lemma 88 and Proposition 119, the corresponding point $\tau_\gamma(t)$ is a bottleneck. However, in Proposition 119, the geodesic γ may intersect the boundaries of the initial and terminal quadrants arbitrarily far from the outermost transverse rectangle R. Hence, there is no upper bound on the length of $\gamma([u, v])$. We now show how to truncate R to achieve a *tame* bottleneck, i.e. a bottleneck where there is a bound on the length of the path $\tau_\gamma([u, v])$ between the initial and terminal quadrants. The measure of the sides of the truncated rectangle will be a definite proportion of the measure of the sides of R.

Definition 120. We say a rectangle $R^0 \subset R$ is ϵ-*nested* inside R, if the leaves containing the sides of R^0 divide R into subrectangles R^i, all of which have measures of sides bounded below as follows:

$$dx(R^i) \geqslant \epsilon \; dx(R) \text{ and } dy(R^i) \geqslant \epsilon \; dy(R).$$

Obviously, Definition 120 needs $\epsilon < 1/3$ and our eventual choice in Proposition 122 is smaller.

Definition 121. We say a rectangle $R^0 \subset R$ is an ϵ-*truncated rectangle*, if it has the following properties.

1. The rectangle R^0 has the same optimal height as R.
2. The rectangle R^0 is ϵ-nested inside R.

By definition the measure of the truncated rectangle is bounded below in terms of the measure of the original rectangle. In particular, $dx(R^0)dy(R^0) \geqslant \epsilon^2 dx(R)dy(R)$. We will label the sides of R^0 using the same convention as the sides for R, using superscripts to distinguish them, i.e. the sides in Λ_+ are α^0_+ and β^0_+, and the sides in Λ_- are α^0_- and β^0_-. This is illustrated in Figure 13.

Proposition 122. *Let (S_h, Λ) be a choice of hyperbolic metric and suited pair of a laminations. Then there are constants $\theta_\Lambda > 0$ and $\epsilon > 0$ such that for any point of intersection $\gamma(t)$ between a non-exceptional geodesic γ and leaf $\ell \in \Lambda_+$ of angle $\theta \leqslant \theta_\Lambda$, and any corresponding outermost rectangle R, there is an ϵ-truncated rectangle $R^0 \subset R$ such that the segment of ℓ lying between α^0_- and β^0_- is $\ell([-r_1, r_2])$, where $r_i = \frac{1}{2} \log \frac{1}{\theta_\Lambda}$, up to additive error at most $\frac{1}{6} \log \frac{1}{\theta_\Lambda} - c_\Lambda$, where here ℓ has a unit speed parametrization such that $\ell(0)$ is the intersection point with γ.*

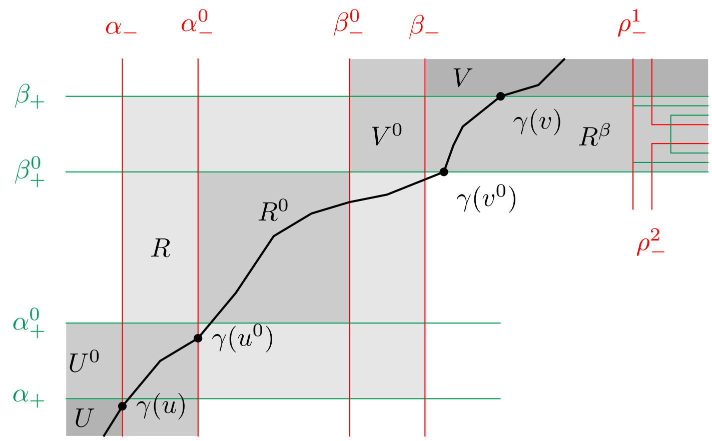

Figure 13: A truncated rectangle at optimal height.

Proof. Suppose that $\gamma(t)$ is an intersection point between γ and $\ell_+ \in \Lambda_+$ of angle $\theta \leqslant \theta_\Lambda$, and let R be the corresponding outermost rectangle. We shall choose $\epsilon = 1/(18Q_\Lambda^2) > 0$, which only depends on (S_h, Λ).

We now construct a smaller rectangle R_0 strictly contained inside R. We shall label the sides of R and R^0 using the notation from Figure 13. We first choose the sides α_-^0 and β_-^0 of R_0 in Λ_-. As before, give ℓ a unit speed parametrization with the intersection point being $\ell(0) = \gamma(t)$. Let $a < a^0 < 0 < b^0 < b$ be the parameters giving the intersections of ℓ with the sides of R and R^0, i.e. $\ell(a) = \ell \cap \alpha_-$, $\ell(a^0) = \ell \cap \alpha_-^0$, $\ell(b^0) = \ell \cap \beta_-^0$ and $\ell(b) = \ell \cap \beta_-$.

Choose intervals $I_1 = \ell(-T - L_\Lambda, -T)$ and $I_2 = \ell(T, T + L_\Lambda)$, where $T = \frac{1}{2}\log\frac{1}{\theta}$, where L_Λ is the constant from Proposition (11.3). Let α_-^0 be the last leaf of intersection of I_1 with Λ_-, and let β_-^0 be the first leaf of intersection of I_2 with Λ_-. Then the segment $\ell([a^0, b^0])$ between α_-^0 and β_-^0 is of the form $\ell(-r_1, r_2)$, where $r_i = \frac{1}{2}\log\frac{1}{\theta}$, up to additive error L_Λ. The required bound on the additive error follows as $L_\Lambda \leqslant \frac{1}{6}\log\frac{1}{\theta_\Lambda} - c_\Lambda$, by our choice of θ_Λ from Definition 109.

We now verify that we may construct a rectangle $R^0 \subset R$ which is ϵ-truncated. Let $\ell([a, a^0])$ be the segment of ℓ between α_- and α_-^0. The hyperbolic length of $\ell([a, a^0])$ is bounded below by

$$\text{length}_{S_h}(\ell([a, a^0]) \geqslant \tfrac{1}{2}\log\tfrac{1}{\theta} - \tfrac{1}{3}\log\tfrac{1}{\theta_\Lambda} + 2c_\Lambda,$$

and so using the quasi-isometry between the hyperbolic and Cannon-Thurston metrics, the measure of this segment is at least

$$dy(\ell([a, a^0])) \geqslant \tfrac{1}{Q_\Lambda}(\tfrac{1}{2}\log\tfrac{1}{\theta} - \tfrac{1}{3}\log\tfrac{1}{\theta_\Lambda} + c_\Lambda),$$

which is positive as $\theta \geqslant \theta_\Lambda$. Using the upper bound on $dy(R)$ from Proposition (119.2), the ratio of the measure of this segment and the measure of $dy(R)$ is then bounded by

$$\frac{dy(\ell([a, a^0]))}{dy(R)} \geqslant \frac{\frac{1}{Q_\Lambda}(\frac{1}{2}\log\frac{1}{\theta} - \frac{1}{3}\log\frac{1}{\theta_\Lambda} + c_\Lambda)}{2Q_\Lambda(\log\frac{1}{\theta} + \frac{1}{2}\log\frac{1}{\theta_\Lambda})} \geqslant \frac{1}{18Q_\Lambda^2} = \epsilon > 0,$$

where the right hand inequality follows as $Q_\Lambda \geqslant 1$. The same argument applies to the segment $\ell([b^0, b])$ between β_-^0 and β_-.

Finally, a lower bound on the ratio $dy(R^0)/dy(R)$ is given by

$$\frac{dy(R^0)}{dy(R)} \geqslant \frac{\frac{1}{Q_\Lambda}(\log\frac{1}{\theta} - \frac{1}{3}\log\frac{1}{\theta_\Lambda} + c_\Lambda)}{2Q_\Lambda(\log\frac{1}{\theta} + \frac{1}{2}\log\frac{1}{\theta_\Lambda})} \geqslant \frac{2}{9Q_\Lambda^2} = 4\epsilon > \epsilon > 0,$$

and as the measures of the three segments $\ell([a, a^0])$, $\ell([a^0, b^0])$ and $\ell([b^0, b])$ sum to $dy(R)$, we obtain

$$0 < 4\epsilon \leqslant \frac{dy(R^0)}{dy(R)} \leqslant 1 - 2\epsilon < 1. \tag{27}$$

We may choose the other two sides α_+^0 and β_+^0 of R_0 such that the measure $dx(R^0) = dy(R^0)dx(R)/dy(R)$, so both R and R^0 have the same optimal height. Furthermore, we may choose the sides so that the leaf of Λ_+ that bisects R in terms of measure, also bisects R^0 in terms of measure. In particular, the bounds on the ratios of the measures of $dy(R^0)/dy(R)$ from (27) applies to the ratio of the measures $dx(R^0)/dx(R)$. As R^0 is centered symmetrically in R with respect to the measure dx, each of the complementary rectangles formed from the intersections of α_+^0 and β_+^0 with R have sides with dx-measure ratio at least ϵ, as required. □

6.3.4 Small angles create tame bottlenecks

In this section, we show that the truncated rectangles give rise to tame bottlenecks.

Corollary 123. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are positive constants $\theta_\Lambda > 0$ (from Definition 109), $r > 0$ and $K \geqslant 0$ such that for any non-exceptional geodesic γ in S_h with unit speed parametrization $\gamma(t)$, if γ intersects a leaf ℓ at $\gamma(t)$ with angle $\theta \leqslant \theta_\Lambda$, then there are parameters $u < t < v$ such that the segment of the test path $\tau_\gamma([u, v])$ is an (r, K)-bottleneck for the ladders over $\gamma((-\infty, u])$ and $\gamma([v, \infty))$, and furthermore, the length of the test path $\tau_\gamma([u, v])$ is at most K.*

Except for the bound on the arc length of the segment of τ_γ between the bottleneck sets given by the initial and terminal quadrants, all properties in Corollary 123 follow from Proposition 122 and Proposition 119. We derive the arc length bound now.

A truncated rectangle at optimal height is shown in Figure 13. We shall write U^0 and V^0 for the initial and terminal quadrants of R^0. Since $U \subset U^0$ and $V \subset V^0$, it follows that if R is a transverse rectangle for γ, then the truncated rectangle R^0 is also a transverse rectangle for γ. We shall write $\gamma(u^0)$ and $\gamma(v^0)$ for the points at which γ intersects the boundaries of the initial and terminal quadrants U^0 and V^0.

To prove Corollary 123, it therefore suffices to prove that the segment $\gamma_\tau([u^0, v^0])$ has bounded length.

Proposition 124. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. There is a constant θ_Λ such that for any $0 < \epsilon < 1$ there is a constant K such that for any ϵ-truncated outermost rectangle R^0, determined by an intersection point of γ with a leaf of an invariant lamination ℓ of angle at most θ_Λ, the arc length of $\tau_\gamma([u^0, v^0])$ is at most K.*

We first verify Corollary 123 from Proposition 124.

Proof (of Corollary 123). By Proposition 119, there are constants θ_Λ and $A > 0$ such that if $\gamma(t)$ is an intersection point of γ with ℓ of angle $\theta \leqslant \theta_\Lambda$, then there is an outermost rectangle R containing $\gamma(t)$ of area at least A. By Proposition 122, there is an $\epsilon > 0$ such that the rectangle R contains an ϵ-truncated outermost rectangle R_0 of area at least $\epsilon^2 A$ which is transverse to γ, giving an (r, K_1)-bottleneck. By Proposition 124, the segment of the test path $\tau_\gamma(u^0, v^0)$ between the bottleneck sets for R^0 has length at most K_2, where K_2 depends only on (S_h, Λ). The result then follows for $K = K_1 + K_2$. □

The rest of this section contains the proof of Proposition 124, which has the following two steps.

1. We consider $\gamma_z([u^0, v^0])$, the image of $\gamma([u^0, v^0])$ at the optimal height z for the rectangle R^0, and show that the length of this path is bounded.

2. We show that the height function varies by a bounded amount over $\gamma([u^0, v^0])$, so projecting $\tau_\gamma([u^0, v^0])$ to $\gamma_z([u^0, v^0])$ increases length by a bounded factor.

The graph of a monotonic function on the unit interval $f\colon I \to I$ has bounded path length. We will use the following analog of monotonicity for real valued functions obtained from paths on surfaces with a suited pair of measured laminations.

Definition 125. Let $\widetilde{S}_h$ be the universal cover of a closed hyperbolic surface, and let Λ_+ and Λ_- be the lifts of a suited pair of measured laminations to $\widetilde{S}_h$. Let $\ell_+ \in \Lambda_+$ and $\ell_- \in \Lambda_-$ be two leaves which intersect at a point c. Let γ be a path from a point a on ℓ_+ to b on ℓ_- such that the interior of γ is disjoint from the two leaves. We say the path γ is *monotonic* if

- Every leaf of Λ_+ which intersects ℓ_- between c and b intersects γ in exactly one point, and γ intersects no other leaves of Λ_+.
- Every leaf of Λ_- which intersects ℓ_+ between c and a intersects γ in exactly one point, and γ intersects no other leaves of Λ_-.

As the above properties are preserved by the vertical flow, a geodesic segment γ in $\widetilde{S}_h = S_0$ is monotonic with respect to ℓ_+ and ℓ^- if an only if for all z, the path $F_z(\gamma)$ in S_z is monotonic with respect to $F_z(\ell_+)$ and $F_z(\ell_-)$. We now show that any geodesic segment in $\widetilde{S}_h$ with endpoints on intersecting leaves is monotonic.

Proposition 126. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Let γ be a segment of a geodesic in $\widetilde{S}_h$ with endpoints on two intersecting leaves ℓ_+ and ℓ_-. Then γ is monotonic with respect to ℓ_+ and ℓ_-.*

Proof. Suppose that c is a point of intersection of leaves ℓ_+ and ℓ_-. Let a be the endpoint of γ on ℓ_+ and b the endpoint of γ on ℓ_-. For the geodesic triangle with vertices a, b and c, let α be its side along ℓ_+ and β its side along ℓ_-. Any leaf of Λ_+ that intersects β crosses the triangle. Since the leaf cannot intersect α it must intersect γ exactly once. Similarly, any leaf of Λ_- which intersects α intersects γ exactly once. Finally, a leaf of either lamination which intersects γ, may only intersect it once, and so must intersect the side contained in the other lamination. $\square$

We now show that the arc length of a monotonic path is bounded by the lengths of the other two sides of the triangle that it forms with the leaves of intersection.

Proposition 127. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Let γ be a monotonic path in $\widetilde{S}_h$ with respect to the two intersecting leaves ℓ_+ and ℓ_-. Let a and b be the endpoints of γ, and let c be the intersection point of the two leaves. Then for any z,*

$$\textit{length}(\gamma_z) \leqslant d_{S_z}(a, c) + d_{S_z}(c, b),$$

where d_{S_z} is the Cannon-Thurston pseudometric on S_z, and length(γ_z) is the arc length of γ_z in this metric.

Proof. In the triangle with vertices a, b, and c we denote the other two sides as $\alpha = [a, c]$ and $\beta = [b, c]$. By definition, the arc length of γ_z is

$$\text{length}(\gamma_z) = \lim_{|P| \to 0} \sum_{i=0}^{n} d_{S_z}(x_i, x_{i+1}), \tag{28}$$

where the limit is taken over partitions $P = \{a = x_0 < x_1 < \cdots < x_{n+1} = b\}$ with $|P| = \max\{x_{i+1} - x_i\}$.

By Definition 125, the union of intervals $\gamma_z \cap R_z$ over all regions R complementary to both laminations is open and dense in γ_z. Hence, for any partition $P = \{a = x_0 < x_1 < \cdots < x_{n+1} = b\}$ there exists a partition $P' = \{a = x'_0 < x'_1 < \cdots < x'_{n+1} = b\}$ such that $|P'| < 2|P|$, and each x'_i lies in a region R_i complementary to both laminations. In particular, $|P'| \to 0$ as $|P| \to 0$.

By Definition 125 again, the intersection $\gamma_z \cap R_i$ cuts off a single vertex of R_i. Since the pseudometric distances between any pair of points in R_i is zero, we may replace x'_i by this vertex without changing each term in the sum in Equation (28) for the partition P'. In particular, we may now assume that each x'_i is an intersection point of the two invariant laminations.

Up to reversing the orientation of γ, we may assume that the initial point of γ lies on ℓ_+ and the terminal point of γ lies on ℓ_-. Consider a pair of adjacent points x'_i and x'_{i+1}, and let ℓ^i_+ be the leaf of Λ_+ through x'_i, and let ℓ^{i+1}_- be the leaf of Λ_- through x'_{i+1}.

By monotonicity, ℓ^i_+ intersects both γ and ℓ_- exactly once, and separates x'_{i+1} from ℓ^+. Therefore the pair of leaves ℓ^i_+ and ℓ^{i+1}_- intersect. Let c'_i be the point of intersection. By the triangle inequality in the Cannon-Thurston metric, $d_{S_z}(x'_i, x'_{i+1}) \leqslant d_{S_z}(x'_i, c'_i) + d_{S_z}(c'_i, x'_{i+1})$. The Cannon-Thurston distance along a leaf is equal to the measure of the leaf with respect to the other invariant lamination, so

$$\text{length}(\gamma_z) \leqslant \lim_{|P| \to 0} \sum_{i=0}^{n} \left(d_{S_z}(x'_i, c'_i) + d_{S_z}(c'_i, x'_{i+1}) \right) = d_{S_z}(a, c) + d_{S_z}(c, b),$$

as required. □

We now complete Step 1 by showing that the length of $\gamma_z([u^0, v^0])$ in S_z is bounded.

Proposition 128. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant θ_Λ such that for any $0 < \epsilon < 1$ there is a constant K such that for any ϵ-truncated outermost rectangle R^0, with optimal height z, determined by an intersection point of γ with a leaf ℓ of one of the invariant laminations of angle at most θ_Λ, the length of $\gamma_z([u^0, v^0])$ is at most K.*

Proof. Up to switching the laminations, we may assume that $\ell = \ell_+ \in \Lambda_+$. By Definition 121, the outermost transverse rectangle R and the ϵ-truncated rectangle R^0 have the same optimal height, which we shall denote by z. As γ is a geodesic in S_h, by Proposition 126, it is monotonic with respect to any two intersecting sides of the rectangle R. At height z, the measures of the sides of R_z are equal. Since the measure of any rectangle is bounded above by $A_{\max}$, the measure of each side of R_z is at most $\sqrt{A_{\max}}$. Therefore, by Proposition 127, it suffices to show that the distances in S_z from $\gamma_z(u^0)$ to R^0_z and from $\gamma_z(v^0)$ to R^0_z are bounded.

We will follow the notation illustrated in Figure 13. Each endpoint of $\gamma([u^0, v^0])$ may lie in either invariant lamination; for definiteness we have drawn in Figure 13 the case where $\gamma(u^0)$ lies in Λ_- and $\gamma(v^0)$ lies in Λ_+.

Suppose that $\gamma_z(v^0)$ lies in Λ_+, as in Figure 13. We will derive a bound for the distance of $\gamma_z(v^0)$ from R^0_z by using properties of the sides in Λ_+ of the (β_+, β^0_+)-maximal rectangle R^β. Exactly the same argument works for when $\gamma_z(v^0)$ in Λ_-, in which case we use the sides in Λ_- of the (β_-, β^0_-)-maximal rectangle.

Let ρ^1_- be the side of R^β separated from α_- by β_-. By maximality of R^β, ρ^1_- contains a side s^1_- of a non-rectangular polygon P. This is illustrated in Figure 14 above. Note that that shading in Figure 14 is different from the shading in Figure 13; in Figure 14 we have only shaded the rectangle R^β. In a cyclic order on the sides in Λ_- of the non-rectangular polygon P, let ρ^2_- be the leaf of Λ_- such that ρ^2_- contains a side s^2_- adjacent to s^1_-, and ρ^2_- intersects β^0_+. Suppose that γ_z intersects ρ^2_-. Then γ is disjoint from the terminal quadrant V, a contradiction. It follows that along β^0_+, the intersection point $\gamma_z(v^0)$ is before the intersection with ρ^2_-.

The measure of the segment of β^0_+ from ρ^1_- to ρ^2_- is zero, so the distance in S_z from R^0_z to $\gamma_z(v^0)$ is at most the horizontal measure of R^β. There is an upper bound $A_{\max}$ on the measure of any rectangle, so

$$dx(R^\beta_z)dy(R^\beta_z) \leqslant A_{\max}. \tag{29}$$

As R^0 is ϵ-truncated with respect to R, $dy(R^1) \geqslant \epsilon dy(R_z)$. Since $dy(R^\beta) = dy(R^1)$, we have $dy(R^\beta) \geqslant \epsilon dy(R_z)$. At the optimal height $dx(R_z) = dy(R_z)$, and by Proposition 81, the area of R_z is at least A_Λ. Hence $dy(R_z)^2 \geqslant A_\Lambda$. It follows that

$$dy(R^\beta_z) \geqslant \epsilon \sqrt{A_\Lambda}. \tag{30}$$

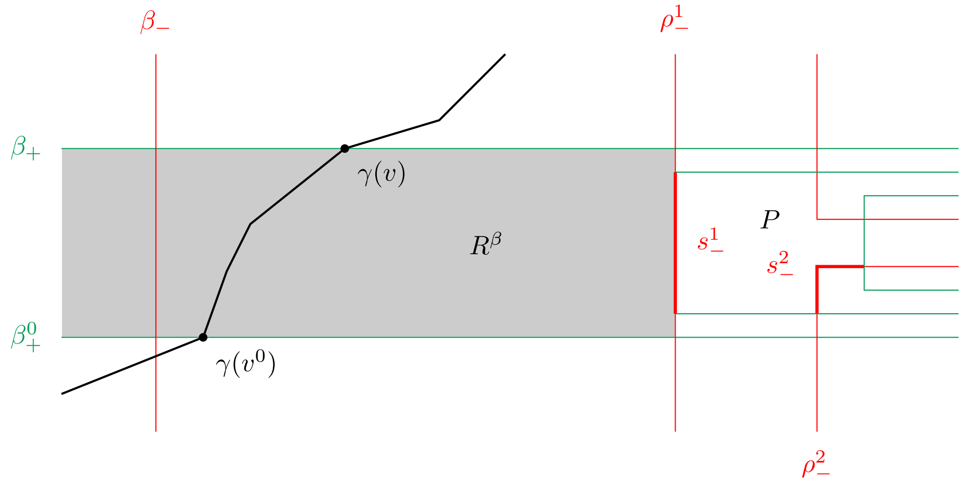

Figure 14: One side of the (β_+, β_+^-)-maximal rectangle R^β from Figure 13.

Combining (29) and (30) gives the following upper bound on the measure of the other side of $R_z^{(1)}$,

$$dx(R_z^\beta) \leqslant \frac{A_{\max}}{\epsilon\sqrt{A_\Lambda}}.$$

For convenience, set $K_1 = A_{\max}/(\epsilon\sqrt{A_\Lambda})$, which only depends on $(S_h, \Lambda), \theta_\Lambda$ and ϵ.

As α^0_- also intersects R^β, this gives the same bound on the distance from $\gamma_z(v^0)$ to α^0_-, i.e.

$$d_{S_z}(\alpha^0_-, \gamma_z(v^0)) \leqslant K_1.$$

By reversing the orientation on γ, the same bounds holds for the distance in S_z from R^0 to $\gamma_z(u^0)$, i.e.

$$d_{S_z}(\beta^0_+, \gamma_z(u^0)) \leqslant K_1.$$

As $\gamma_z([u^0, v^0])$ is monotonic in S_0, it is also monotonic in S_z, so by Proposition 127, the arc length of $\gamma_z([u^0, v^0])$ is at most

$$\text{length}(\gamma_z([u^0, v^0])) \leqslant d_{S_z}(\alpha^0_-, \gamma_z(v^0)) + d_{S_z}(\beta^0_+, \gamma_z(u^0)) \leqslant 2K_1,$$

which ony depends on $(S_h, \Lambda), \theta_\Lambda$ and ϵ, as required. □

We now finish Step 2 by showing that the height function changes by a bounded amount along $\gamma([u^0, v^0])$.

Proposition 129. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. There is a constant θ_Λ such that for any $0 < \epsilon < 1$ there is a constant K such that for any ϵ-truncated outermost rectangle R^0, with optimal height z, determined by an intersection point of γ with a leaf ℓ of an invariant lamination of angle $\theta \leqslant \theta_\Lambda$, for any $t \in [u^0, v^0]$, the value of the height function $h_\gamma(t)$ is equal to $\log_k \log \frac{1}{\theta}$ up to additive error at most K.*

Up to swapping the laminations, we may assume the leaf of intersection $\ell = \ell_+$ lies in Λ_+.

We prove Proposition 129 in two parts. In Proposition 130 we bound the change in the height function along the segment of γ_z lying between the sides α^0_- and β^0_- of R^0. Note that if both endpoints of $\gamma([u^0, v^0])$ lie in Λ_- then $\gamma([u^0, v^0])$ equals this segment and hence Proposition 129 follows.

So we may suppose that one of the endpoints of $\gamma([u^0, v^0])$ is in Λ_+. In this case, we use a different set of estimates in S_h to bound the change in the height function in Proposition 131. This covers all cases and so Proposition 129 is an immediate consequence. The constants labelled K constructed in each of Proposition 130 and Proposition 131 may be different, but we can just take their maximum for Proposition 129.

Proposition 130. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. There is a constant θ_Λ such that for any $0 < \epsilon < 1$ there is a constant K such that for any ϵ-truncated outermost rectangle R^0, with optimal height z, determined by an intersection point of γ with a leaf ℓ of an invariant lamination of angle $\theta \leqslant \theta_\Lambda$, for any t such that $\gamma(t)$ lies between the sides α^0_- and β^0_- of R^0 which lie in Λ_-, the value of the height function $h_\gamma(t)$ is equal to $\log_k \log \frac{1}{\theta}$ up to additive error at most K.*

Proof. We use the notation in Figure 13. Up to swapping the laminations, we may assume that γ makes angle at most θ_Λ at an intersection point $\gamma(t)$ with a leaf $\ell_+ \in \Lambda_+$. Let R^0 be the corresponding ϵ-truncated outermost rectangle. We denote the segment of γ between the sides α^0_- and β^0_- by $\gamma([a, b])$, as illustrated in Figure 15. We will estimate the change in the height function along $\gamma([a, b])$.

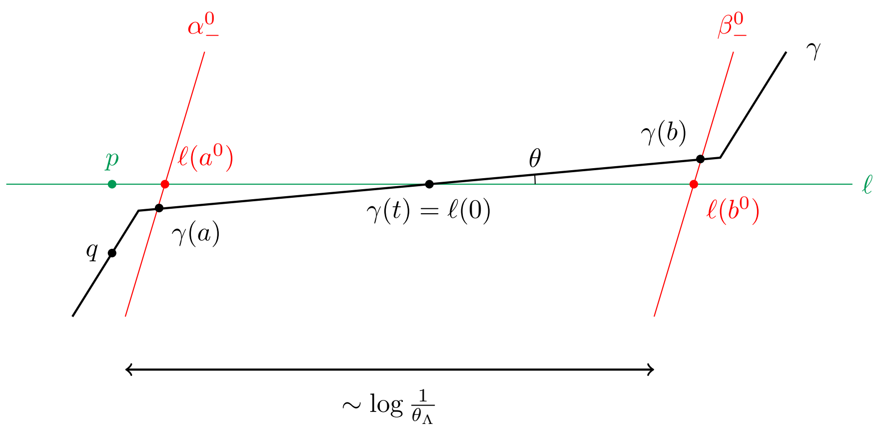

Figure 15: Estimating the length of γ between the sides of a truncated rectangle.

By Proposition 108, the value of the radius function at the intersection point $\gamma(t)$ is roughly equal to $\log \frac{1}{\theta}$. More precisely, using (16) from Proposition 108,

$$\rho_{\gamma,\ell}(t) \leqslant \rho_{\gamma,\Lambda_+}(t) \leqslant \rho_{\gamma,\ell}(t) + K_1,$$

where K_1 is the constant from Proposition 108, which depends only on (S_h, Λ). Using the definition of the radius function (13), gives

$$\log \tfrac{1}{\theta} \leqslant \rho_{\gamma,\Lambda_+}(t) \leqslant \log \tfrac{1}{\theta} + K_1.$$

We now find an upper bound for the hyperbolic length of the segment $\gamma[a, t]$. The same argument will give an upper bound for the hyperbolic length of $\gamma([t, b])$. Let $\ell(a^0)$ be the point of intersection between ℓ and α^0_-, and let $\ell(b^0)$ be the point of intersection between ℓ and β^0_-. By Proposition 122, then length of $\ell([a^0, 0])$ is bounded above by

$$\text{length}_{S_h}(\ell([a^0, 0])) \leqslant \tfrac{1}{2} \log \tfrac{1}{\theta} + \tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda} - c_\Lambda. \tag{31}$$

By the triangle inequality,

$$d_{S_h}(\gamma(a), \gamma(t)) \leqslant d_{S_h}(\gamma(a), \ell(a^0)) + d_{S_h}(\ell(a^0), \ell(0)).$$

We now find an upper bound for the hyperbolic distance from $\gamma(a)$ to $\ell(a^0)$. Let p be a point on ℓ distance $2\rho_\Lambda$ from $\ell(a^0)$, such that $\ell(a^0)$ separates p from $\ell(0)$, and let q be the closest point on γ to p.

By Proposition 25 the distance from p to γ is at most θe^t, where $t \leqslant \frac{1}{2}\log\frac{1}{\theta} + \frac{1}{4}\log\frac{1}{\theta_\Lambda} - c_\Lambda + 2\rho_\Lambda$, so

$$d_{S_h}(p,q) \leqslant \theta e^{\frac{1}{2}\log\frac{1}{\theta}+\frac{1}{4}\log\frac{1}{\theta_\Lambda}-c_\Lambda+2\rho_\Lambda},$$

which simplifies to

$$d_{S_h}(p,q) \leqslant \theta^{\frac{1}{2}}\theta_\Lambda^{-\frac{1}{4}}e^{-c_\Lambda+2\rho_\Lambda},$$

and as $\theta \leqslant \theta_\Lambda$,

$$d_{S_h}(p,q) \leqslant \theta_\Lambda^{\frac{1}{4}}e^{-c_\Lambda+2\rho_\Lambda}.$$

By our choice of θ_Λ,

$$d_{S_h}(p,q) \leqslant \tfrac{1}{4}\rho_\Lambda, \tag{32}$$

and so the nearest point projection of q to ℓ lies within distance $\frac{1}{2}\rho_\Lambda$ of p. In particular, it lies outside the nearest point projection interval of α^0_- to ℓ. This means that q lies past $\gamma(a)$ from $\gamma(t)$, and so the distance from $\gamma(t)$ to q is an upper bound on the distance from $\gamma(t)$ to $\gamma(a)$, i.e.

$$d_{S_h}(\gamma(a),\gamma(t)) \leqslant d_{S_h}(q,\gamma(t)),$$

and then by the triangle inequality,

$$d_{S_h}(\gamma(a),\gamma(t)) \leqslant d_{S_h}(q,p) + d_{S_h}(p,\gamma(t)).$$

Using (32), and the fact that $\gamma(t) = \ell(0)$, and $\ell(a^0)$ lies between p and $\ell(0)$, gives

$$d_{S_h}(\gamma(a),\gamma(t)) \leqslant \tfrac{1}{4}\rho_\Lambda + d_{S_h}(p,\ell(a^0)) + d_{S_h}(\ell(a^0),\ell(0)).$$

By our choice of p, the distance from p to $\ell(a^0)$ is $2\rho_\Lambda$. Using the upper bound from (31), we get

$$d_{S_h}(\gamma(a),\gamma(t)) \leqslant \tfrac{1}{2}\log\tfrac{1}{\theta} + \tfrac{1}{6}\log\tfrac{1}{\theta_\Lambda} - c_\Lambda + \tfrac{9}{4}\rho_\Lambda.$$

Our choice of θ_Λ ensures that $\log\frac{1}{\theta_p a} \geqslant \frac{9}{4}\rho_\Lambda$, so we simplify the inequality above to

$$d_{S_h}(\gamma(a),\gamma(t)) \leqslant \tfrac{1}{2}\log\tfrac{1}{\theta} + \tfrac{1}{3}\log\tfrac{1}{\theta_\Lambda}.$$

Similarly, we get exactly the same bound for $d_{S_h}(\gamma(b),\gamma(t))$.

As the radius function is 1-Lipschitz with respect to the hyperbolic metric, for any $t \in [a,b]$ the value of the radius function at $\gamma(t)$ is at least

$$\rho_{\gamma,\Lambda_+}(t) \geqslant \tfrac{1}{2}\log\tfrac{1}{\theta} - \tfrac{1}{3}\log\tfrac{1}{\theta_\Lambda} \geqslant \tfrac{1}{6}\log\tfrac{1}{\theta_\Lambda} > 0, \tag{33}$$

and the value of the radius function at t is at most

$$\rho_{\gamma,\Lambda_+}(t) \leqslant \tfrac{3}{2}\log\tfrac{1}{\theta} + \tfrac{1}{3}\log\tfrac{1}{\theta_\Lambda} + K_1. \tag{34}$$

As the height function is $\log_k$ of a floor function of the radius function, the change in the height funtion is therefore bounded by the logarithm base k of the ratio between the values of the height function along $\gamma([a,b])$. In particular, using (33) and (34) the change in height function along $\gamma([a,b])$ is at most

$$\log_k \frac{\frac{3}{2}\log\frac{1}{\theta} + \frac{1}{3}\log\frac{1}{\theta_\Lambda} + K_1}{\frac{1}{6}\log\frac{1}{\theta_\Lambda}} \leqslant \log_k\left(11\log\tfrac{1}{\theta_\Lambda} + 6K_1\right) = K_2,$$

where the right hand side K_2 only depends on (S_h,Λ), as required. □

We now consider the case in which an endpoint of $\gamma([u^0, v^0])$ lies in Λ_+, and so $\gamma([u^0, v^0])$ is not contained between the sides of R^0 that are in Λ_-.

Proposition 131. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a regular pair of measured laminations. There is a constant θ_Λ such that for any $0 < \epsilon < 1$ there is a constant K such that for any ϵ-truncated outermost rectangle R^0, with optimal height z, determined by an intersection point of γ with a leaf ℓ of an invariant lamination of angle $\theta \leqslant \theta_\Lambda$, for any t such that $\gamma(t)$ lies outside the sides α^0_- and β^0_- of R^0 which lie in Λ_-, the value of the height function $h_\gamma(t)$ is equal to $\log_k \log \frac{1}{\theta}$ up to additive error at most K.*

Proof. Up to reversing the orientation on γ, we may assume that the endpoint of $\gamma([u^0, v^0])$ that lies in Λ_+ is $\gamma(v^0)$, as illustrated in Figure 13.

The radius function ρ_ℓ for ℓ at $\gamma(t)$ has a local maximum of height bounded below by $\log \frac{1}{\theta}$.

Let I_ℓ be the projection interval for ℓ onto γ, and I_{β_+} the nearest point projection interval of β_+ to γ.

The leaves ℓ and β_+ meet α^0_- at a bounded angle, and since α^0_- intersects γ, the nearest point projection interval I_ℓ is coarsely contained in the nearest point projection interval I_{β_+}.

We now find an upper bound on the distance from R^0 to $\gamma(v^0)$. By Proposition 82, the measure of the rectangle R^1 is at most $dx(R^1)dy(R^1) \leqslant A_{\max}$, so

$$dy(R^1) \leqslant \frac{A_{\max}}{dx(R^1)}.$$

The height of the rectangle R^1 is at least $dx(R^1) \geqslant \epsilon dx(R)$ and the measure of R is at least $dx(R)dy(R) \geqslant A_\Lambda$, so

$$dy(R^1) \leqslant \frac{A_{\max}}{\epsilon A_\Lambda} dy(R).$$

Using the bound on the measure of the side of the outermost rectangle from Proposition (119.2) gives

$$dy(R^1) \leqslant \frac{A_{\max}}{\epsilon A_\Lambda} 2Q_\Lambda (\log \tfrac{1}{\theta} + \tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda}).$$

Using the quasi-isometry between the hyperbolic metric and the Cannon-Thurston metric gives

$$d_{S_h}(\beta^0_-, \gamma(v^0)) \leqslant Q_\Lambda \frac{A_{\max}}{\epsilon A_\Lambda} 2Q_\Lambda (\log \tfrac{1}{\theta} + \tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda}) + c_\Lambda.$$

By the triangle inequality, the distance along γ from β^0_- to $\gamma(v^0)$ is at most $d_{S_h}(\beta^0_-, \gamma(v^0))$ plus the distance along β^0_- from β^0_+ to $p = \gamma \cap \beta^0_-$. This latter distance is bounded by the side length of R_0 contained in β^0_-. This gives

$$d_{S_h}(p, \gamma(v^0)) \leqslant Q_\Lambda \frac{A_{\max}}{\epsilon A_\Lambda} 2Q_\Lambda (\log \tfrac{1}{\theta} + \tfrac{1}{6} \log \tfrac{1}{\theta_\Lambda}) + c_\Lambda + \frac{Q_\Lambda A_\Lambda}{\frac{2}{Q_\Lambda}(\log \frac{1}{\theta} - \frac{1}{6} \log \frac{1}{\theta_\Lambda})} + c_\Lambda.$$

Using the fact that $\theta \leqslant \theta_\Lambda$ gives

$$d_{S_h}(p, \gamma(v^0)) \leqslant K_3 \log \tfrac{1}{\theta} + K_4,$$

where K_3 and K_4 depend only on Λ.

The radius function ρ_{β_+} is a coarse lower bound for the radius function ρ_γ, so in particular, $\rho_\gamma(v^0)$ is at most $K_3 \log \frac{1}{\theta} + K_4$. Therefore the height difference between $\gamma(t)$ and $\gamma(v^0)$ is at most

$$\log_k \frac{(K_3 + 1) \log \frac{1}{\theta} + K_4}{\log \frac{1}{\theta}} \leqslant \log_k (K_3 + 1 + K_4),$$

which depends only on (S_h, Λ), as required. □

We finally show that the length of $\tau_\gamma([u^0, v^0])$ is bounded.

Proof (of Proposition 124). By Proposition 110, there is a constant K_1 such that the value of the height function at the intersection point is equal to $\log_k \log \frac{1}{\theta}$ up to additive error at most K_1. By Proposition 129, there is a constant K_2 such that the variation in height along $\gamma([u^0, v^0])$ is at most K_2. Therefore projecting $\tau_\gamma([u^0, v^0])$ to $\gamma_z([u^0, v^0])$ changes the length by at most a factor of $k^{K_1+K_2}$.

By Proposition 128, there is a constant K_3 such that $\gamma_z([u^0, v^0])$ has arc length at most K_3. Hence, $\gamma_\tau([u^0, v^0])$ has arc length at most $k^{K_1+K_2}K_3$, where all the constants depend only on (S_h, Λ), as required. □

6.3.5 Corner segments create tame bottlenecks

By Definition 83, a *corner segment* is be a segment of γ that cuts off a corner of an innermost rectangle. For some terminology, we now distinguish between two possibilities for the corner segments.

Definition 132. A segment $\gamma(I)$ of a non-exceptional geodesic is a *positive length corner segment* if it is a corner segment with positive length. A segment $\gamma(I)$ is a *zero length corner segment* if I consists of a single point, and $\gamma(I)$ is the vertex of an innermost rectangle.

In either case, a corner segment determines exactly two angles of intersection, one with each invariant lamination. In this section, we will show that a corner segment creates a bottleneck in $\widetilde{S}_h \times \mathbb{R}$.

Lemma 100. Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants r and K, such that for any corner segment $I = [t_1, t_2]$ of any non-exceptional geodesic γ, there are parameters $u \leqslant t_1 \leqslant t_2 \leqslant v$ such that for any $t \in [u, v]$, the point $\tau_\gamma(t)$ is a (r, K)-bottleneck for the ladders over $\gamma((-\infty, u])$ and $\gamma([v, \infty))$. Furthermore, the length of $\tau_\gamma([u, v])$ is at most K.

If at least one angle is small, then it suffices to observe that the corner segment is contained in the interval $[u, v]$ in Corollary 123, and Lemma 100 follows.

It therefore remains to show that if a corner segment meets both invariant laminations with large angles, then it creates a bottleneck.

Proposition 133. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Let θ_Λ be the constant from Definition 109. Then there are constants $r > 0$ and $K \geqslant 0$ such that for any non-exceptional geodesic γ in S_h, for any corner segment $\gamma([t_1, t_2])$, meeting both invariant laminations at angles at least θ_Λ, there are parameters $u \leqslant t_1 \leqslant t_2 \leqslant v$ such that for any $t \in [u, v]$, the point $\tau_\gamma(t)$ is a (r, K)-bottleneck for the ladders over $\gamma((-\infty, u])$ and $\gamma([v, \infty))$. Furthermore, the length of $\tau_\gamma([u, v])$ is at most K.*

We start by showing that any geodesic which intersects a leaf of a lamination intersects a rectangle intersecting that leaf, with a lower bound on its transverse measure, which depends only on the angle of intersection. The final result will then follow from constructing these rectangles for both leaves at each end of the corner segment, and then taking the intersection of these two rectangles.

Proposition 134. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then for any non-exceptional geodesic γ intersecting a leaf of a lamination $\ell \in \Lambda_+$ at angle $0 < \theta \leqslant \pi/2$, there are leaves ℓ_1 and ℓ_2 in the other lamination Λ_- with the following properties.*

1. *The leaf ℓ is a common leaf of intersection for ℓ_1 and ℓ_2.*
2. *The nearest point projections of γ, ℓ_1 and ℓ_2 to ℓ are all disjoint.*
3. *The (ℓ_1, ℓ_2)-maximal rectangle R intersects both ℓ and γ.*

4. The transverse measure of the rectangle is at least

$$dx(R) \geqslant \frac{A_\Lambda}{2Q_\Lambda(\log\frac{1}{\theta} + \log\frac{1}{\alpha} + 2T_0 + L_\Lambda) + c_\Lambda},$$

where Q_Λ and c_Λ are the constants defined in Proposition 38, T_0 is the constant defined in Proposition 25, and L_Λ is the constant defined in Proposition (11.3), and all these constants depend only on the choice of (S_h, Λ).

The same result holds with the invariant measured laminations (Λ_+, dx) and (Λ_-, dy) swapped.

Proof. Up to swapping the laminations we may assume that ℓ is in Λ_+.

We parametrize ℓ with a unit speed such that the intersection point with γ is $\ell(0)$. Then the nearest point projection of γ to ℓ is contained in the interval $I = \ell(-\log\frac{1}{\theta} - T_0, \log\frac{1}{\theta} + T_0)$.

Consider intervals I_1 and I_2 of length L_Λ on either side of I, distance $\log\frac{1}{\alpha} + T_0$ from I. Each interval I_i intersects a leaf ℓ_-^i of Λ_-, whose nearest point projection to ℓ is disjoint from I. The leaf ℓ is then common for both ℓ_-^1 and ℓ_-^2, so there is a (ℓ_-^1, ℓ_-^2)-maximal rectangle R with area at least A_Λ. As the measure of the side of the rectangle parallel to ℓ is at most $2Q_\Lambda(\log\frac{1}{\theta} + \log\frac{1}{\alpha} + 2T_0 + L_\Lambda) + c_\Lambda$, the result follows. □

We now complete the proof of Proposition 133.

Proof (of Proposition 133). Let ℓ^+ and ℓ^- be the leaves of intersection at each end of the corner segment. Up to reversing the orientation on γ, we may assume that γ hits ℓ^- first, as illustrated in Figure 16.

By Proposition 134, the intersection point $\gamma \cap \ell^-$ is contained in a rectangle R_1 with measure

$$dy(R_1) \geqslant \frac{A_\Lambda}{2Q_\Lambda(\log\frac{1}{\theta_\Lambda} + \log\frac{1}{\alpha} + 2T_0 + L_{|}pa) + c_\Lambda} := K_1.$$

such that the Λ_+ sides of R_1 have nearest point projections to ℓ^- disjoint from the nearest point projection of γ to ℓ^-. In particular, ℓ^+ intersects the interior of R_1, and the sides of R_1 in Λ_- intersect γ.

Similarly by Proposition 134, the other intersection point $\gamma \cap \ell^+$ is contained in a rectangle R_2 with measure $dx(R_2) \geqslant K_1$ such that ℓ^- intersects the interior of R_2, and the sides of R_2 in Λ_+ intersect γ.

The intersection $R = R_1 \cap R_2$ is thus a rectangle such that R contains the point of intersection of ℓ^+ and ℓ^-, the measure of R satisfies $dx(R)dy(R) \geqslant K_1^2$, and all sides of R intersect γ. Hence R is a transverse rectangle for γ. We use the previous notation for the sides of R, so the initial side of R in Λ_- is α^-, and the terminal side of R in Λ_+ is β^+.

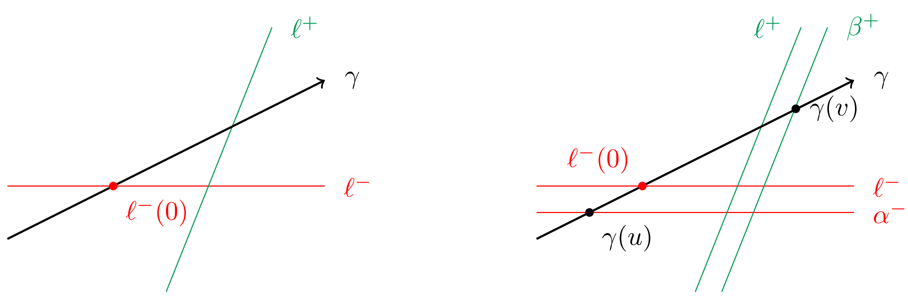

Figure 16: A corner segment with large angles.

Let U_R and V_R be the initial and terminal quadrants for R, and choose u and v such that $U = U_R \cap F(\gamma) = F(\gamma((-\infty, u]))$ and $V = V_R \cap F(\gamma) = F(\gamma([v, \infty)))$. By Lemma 88 there are constants r and K_2 such that any point $t \in [u, v]$ is a (r, K_2)-bottleneck for U and V.

We now bound the length of $\gamma([u, v])$.

Let p be the intersection point between α^- and β^+. As $\gamma([u, v])$ is monotonic with respect to α^- and β^+, by Proposition 127 the length of $\gamma([u, v])$ is bounded by $d(\gamma(u), p) + d(p, \gamma(v))$.

Let q be the nearest point projection of $\gamma(u)$ to ℓ^-. The point $\gamma(u)$ lies on γ, and so is contained in the nearest point projection of γ to ℓ^-.

By construction, the radius of the nearest point projection interval of α^- to ℓ is at least $\log \frac{1}{\alpha} + T_0$, and q is at least distance $\log \frac{1}{\alpha} + T_0$ from the endpoints of the projection interval. Therefore the distance in $PSL(2, \mathbb{R})$ between $\gamma(u)$ and q is at most α, so the distance in $\mathbb{H}^2$ is at most $\alpha + \pi$.

Therefore $d(\gamma(u), p) \leqslant \log \frac{1}{\theta_\Lambda} + T_0 + \alpha + \pi$.

Therefore the length of $\gamma([u, v])$ is at most $K_3 := 2(\log \frac{1}{\theta_\Lambda} + T_0 + \alpha + \pi)$.

We now bound the value of the height function along the corner segment I. As $\gamma(I)$ meets Λ_- at angle at least θ_Λ at t_1, the height function at t_1 is coarsely non-positive, i.e. $h_\gamma(t_1) \leqslant K_4$, where K_4 is the constant from Proposition 110. Similarly, as $\gamma(I)$ meets Λ_+ at angle at least θ_Λ at t_2, the height function at t_2 is coarsely non-negative, i.e. $h_\gamma(t_1) \geqslant -K_4$. By Proposition 114, the height function is $(1/\log k)$-Lipschitz, and the length of I is at most L_Λ, where L_Λ is the constant from Proposition (11.3). Therefore, for any $t \in I$, the value of the height function is bounded above and below, i.e. $|h_\gamma(t)| \leqslant K_4 + L_\Lambda / \log k =: K_5$.

The lengths of $\gamma([u, t_1])$ and $\gamma([t_2, u])$ are at most K_3, and the height function is $(1/\log k)$-Lipschitz. Therefore, the value of the height function on $[u, v]$ is at most $|h_\gamma(t)| \leqslant K_5 + K_3 / \log k =: K_6$.

Nearest point projection of τ_γ to S_0 therefore distorts distance by at most a factor of k^{K_6}.

Therefore the length of $\tau_\gamma([u, v])$ is at most $K_3 k^{K_6}$. □

6.4 Straight segments are quasigeodesic

In this section we prove that segments of the test path over straight segments are quasigeodesic.

Lemma 101. Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants Q and c such that for any non-exceptional geodesic γ, and any straight interval I, the test path $\tau_\gamma(I)$ is (Q, c)-quasigeodesic.

In Section 6.4.1, we show that if the intersection interval of γ with an ideal complementary region R is short, then the test path over that interval is also short. We may then consider segments $\gamma \cap R$ which are reasonably long. In Section 6.4.2, we show that if $\gamma \cap R$ is sufficiently long, then the height function does not change sign along this segment. Inside the flow set $F(R)$, the Cannon-Thurston metric is quasi-isometric to the union of a number of hyperbolic halfspaces glued along a common bi-infinite boundary geodesic. In Section 6.4.3, we show that specific paths in these halfspaces are quasigeodesic. In Section 6.4.4, we show that the test path over a long segment $\gamma \cap R$ is close to one of these quasigeodesic paths, and is hence quasigeodesic. We then complete the final step to show that this also applies to a straight segment that is a union of two intersection segments meeting in a non-rectangular polygon.

6.4.1 Short segments in complementary regions

In this section, we show that segments of bounded length in ideal complementary regions give rise to bounded length segments of the test path.

Proposition 135. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then for any constant $L > 0$ there is a constant $K > 0$ such that for any non-exceptional geodesic γ in $\widetilde{S}_h$ and for any ideal complementary region R, if the length of the intersection interval I_R is at most L, then the length of the test path $\tau_\gamma(I_R)$ over that interval is at most K.*

Proof. Up to swapping laminations and reversing the sign on the height function, we may assume that the complementary region R has boundary in Λ_+.

The interior of the intersection interval I_R is disjoint from Λ_+. By the quasi-isometry between the hyperbolic and Cannon–Thurston metrics (Proposition 38), the Λ_- measure of I_R is at most $Q_\Lambda L + c_\Lambda$.

If the interior of I_R is disjoint from both laminations, then I_R is an innermost segment. By Corollary 113, the arc length of $\tau_\gamma(I_R)$ is then at most K_1. So we may assume that the intersection interval I_R intersects Λ_-. The metric on the ladder $F(\gamma(I_R))$ is then determined by the z-coordinate, and the measure of the intersections with Λ_-.

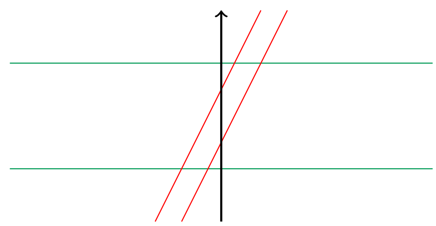

Figure 17: Outermost transverse rectangles determined by ℓ_1^+ and ℓ_2^+.

Moving in the positive z-direction scales the Λ_--measure by a factor of k^{-z}. Thus if the height function along I_R is bounded below by $L_1 = -2L - L/\log k$, then the arc length of $\tau_\gamma(I_R)$ in the Cannon-Thurston metric is at most $(Q_\Lambda L + c_\Lambda)k^{L_1}$.

Now suppose there is a point t on I_R with height function $h_\gamma(t)$ less than L_1. As the height function is $(1/\log k)$-Lipschitz, this implies that the height function is at most $-2L$ on all of I_R.

Let ℓ_1^- and ℓ_2^- be outermost leaves of Λ_- intersecting I_R at $\gamma(t_1)$ and $\gamma(t_2)$ with angles θ_1 and θ_2. By Proposition 110, the height function determines the angle up to bounded error, so both θ_1 and θ_2 are at most $\theta_\Lambda e^{-k^{2L-K_1}}$, where K_1 is the constant from Proposition 110.

Consider the maximal rectangle with sides contained in ℓ_1 and ℓ_2. These two sides fellow travel for distance at least $\log \frac{1}{\theta_1}$, up to additive error.

Therefore by Proposition 82 the measure of the other sides is at most $A_{\max}/\log \frac{1}{\theta_1}$.

As the height function is $(1/\log k)$-Lipshitz, the length of $\tau_\gamma(I_R)$ is at most $L/\log k + k^{A_{\max}/\log \frac{1}{\theta_1}} \leqslant L/\log k + k^{A_{\max}/\log \frac{1}{\theta_\Lambda}}$. ☐

6.4.2 Long intersection intervals have height functions with consistent signs

In this section, we show that if R_+ is an ideal complementary region of Λ^+, and if the length of the intersection interval is sufficiently long, then the height function along the intersection interval is non-negative. Swapping laminations a similar statement holds for ideal complementary regions of Λ_-.

Proposition 136. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there is a constant L_R, such that for any ideal polygon R with boundary in Λ_+, and any non-exceptional geodesic γ crossing R in an intersection interval I_R of length at least L_R, the height function along the intersection interval I_R satisfies $h_\gamma(t) \geqslant 0$. Similarly, if R has ideal boundary contained in Λ_-, then the height function along I_R satisfies $h_\gamma(t) \leqslant 0$.*

Proof. Up to swapping laminations and reversing the sign of the height function, we may assume that R is an ideal complementary region of Λ_+. We choose $L_R > 2D_\Lambda + 4L_\Lambda + 2\rho_\Lambda$, where D_Λ is the constant from Proposition 79, L_Λ is the constant from Proposition (11.3) and ρ_Λ is the constant from Proposition 26. We denote $\gamma \cap R$ by $\gamma(I_R)$ and assume that the interval $\gamma(I_R)$ has length at least L_R. Suppose that the interior of $\gamma(I_R)$ is disjoint from Λ_-. Then $\gamma(I_R)$ is disjoint from both laminations. By Proposition 79, it is an innermost interval with length at most D_Λ which is not possible as $L_R > D_\Lambda$. Thus $\gamma(I_R)$ intersects Λ_-.

Suppose that $\gamma(t_1)$ is a point on $\gamma(I_R)$ such that $h_\gamma(t_1) < 0$. Since extended laminations are closed, by Definition 94 there is a leaf ℓ_1^- of $\overline{\Lambda}_-$ with distance in $\mathrm{PSL}(2,\mathbb{R})$ at most θ_Λ from $\gamma(t_1)$.

Up to reversing the orientation of γ, let $t_2 \geqslant t_1$ be the smallest time such that there is a leaf ℓ_2^- of Λ^- intersecting $\gamma(I_R)$ at the point $\gamma(t_2)$. If $t_2 \neq t_1$ then ℓ_2^- is a boundary leaf of an ideal complementary region R' of Λ_- and the segment $\gamma(t_1, t_2)$ is contained in R'. It follows that $t_2 - t_1 < D_\Lambda$. By Proposition 111, the radius function is 1-Lipschitz. Therefore, the difference between the values $\rho_\gamma(t_1)$ and $\rho_\gamma(t_2)$ of the radius function is at most D_Λ. Therefore the radius of the projection interval for ℓ_2^- is at least $\log \frac{1}{\theta_\Lambda} - D_\Lambda$.

There is a boundary leaf ℓ^+ of R that intersects ℓ_2^- within distance L_Λ of $\gamma(t_2)$. Let the intersection point be p, and let q be the nearest point on γ to p. We now show that ℓ^+ intersects γ, as illustrated in Figure 18, where we have drawn ℓ^+ intersecting γ at $\gamma(t_3)$.

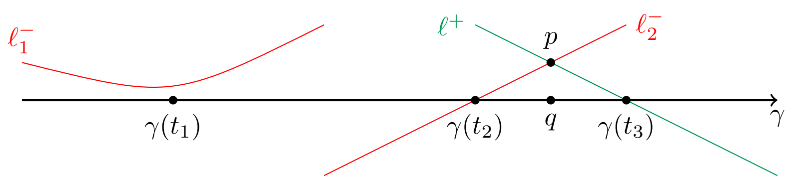

Figure 18: An intersection point close to a negative value of the height function.

Suppose ℓ^+ does not intersect γ. Then one of the endpoints of ℓ^+ lies between the endpoints of ℓ_2^- and γ_+ on the boundary circle at infinity. This means that the projection interval for ℓ^+ onto γ must extend past the end of the projection interval for ℓ_2^- onto γ in the direction of the endpoint γ_+. The distance from $\gamma(t_2)$ to q is at most $2L_\Lambda$. Thus the length of the overlap of the projection intervals for ℓ_2^- and ℓ^+ to γ is at least $\log \frac{1}{\theta_\Lambda} - D_\Lambda - 2L_\Lambda$, which by our choice of θ_Λ is greater than ρ_Λ, the maximal overlap between projections of leaves to γ, contradicting Proposition 26.

We conclude that ℓ^+ intersects $\gamma(I_R)$ at a point $\gamma(t_3)$. The point $\gamma(t_3)$ lies in the nearest point projection interval of ℓ^+ to γ. So the distance from q to $\gamma(t_3)$ is at most ρ_Λ. Therefore the distance from $\gamma(t_1)$ to $\gamma(t_3)$ is at most $D_\Lambda + 2L_\Lambda + \rho_\Lambda$. As this argument applies to traveling along γ in the other direction, this shows that the length of I_R is at most $2(D_\Lambda + 2L_\Lambda + \rho_\Lambda)$ which is less than L_R, a contradiction. □

6.4.3 Quasigeodesic paths in the upper half space model

Recall that the metric on the ladder $F(\ell_-)$ is not the standard metric on the upper half space but is quasi-isometric to it under the map $(x, z) \mapsto (x, k^{-z})$.

In this section, we show that paths in the upper half space arising as graphs of 1-Lipschitz functions $\mathbb{R} \to \mathbb{R}_+$ are quasi-geodesic. To begin with, a line with slope one in the upper half space model stays a constant distance from a vertical line, and is hence a quasigeodesic. This is illustrated in both models in Figure 19. From this we will deduce that absolute value functions are quasi-geodesic. We will then show that 1-Lipschitz functions are contained in bounded neighborhoods of certain absolute value functions, to get the required conclusion.

In subsequent sections, we will show that the height function over a single leaf is contained in the above class of functions. Hence, a test path in the upper half space model is a quasigeodesic.

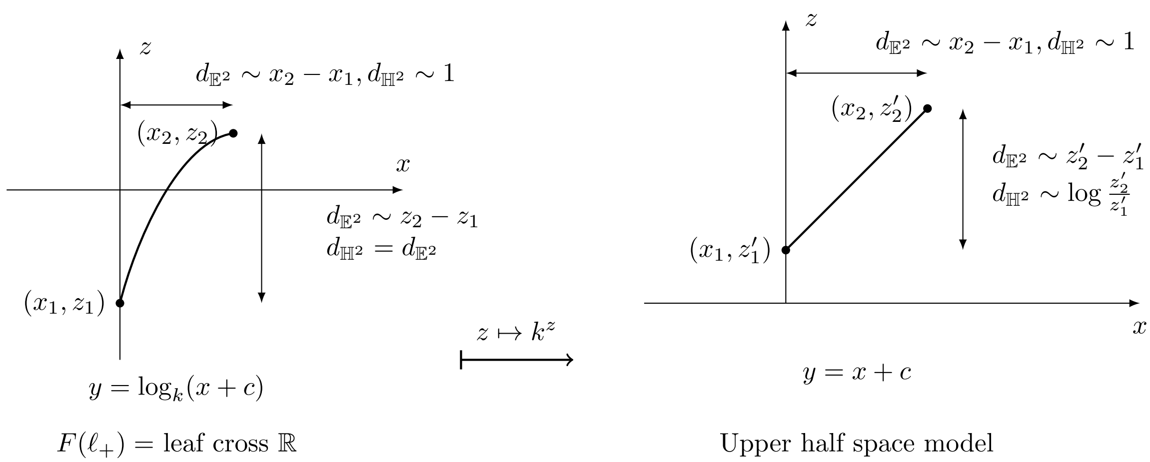

Figure 19: Quasigeodesics in two models for $\mathbb{H}^2$.

Definition 137. Suppose that $I \subseteq \mathbb{R}$ is a (possibly infinite) subinterval of $\mathbb{R}$. Suppose that $a\colon I \to \mathbb{R}_+$ is a function. We call the path α in the upper half space model of $\mathbb{H}^2$ given by $\alpha(t) = (t, a(t))$ *the path determined by the function* a.

We consider a specific collection of paths determined by absolute value functions.

Definition 138. Given constants $h > 1$ and x, an *absolute value path* is the path in the upper half space determined by $a(t) = h - |t - x|$ defined on the interval $I = |t - x| \leqslant h - 1$. We have chosen I so that $a(t) \geqslant 1$ for all $t \in I$.

We start by showing that the paths determined by these functions are quasigeodesic.

Proposition 139. *There are constants $Q \geqslant 1$ and $c \geqslant 0$ such that every absolute value path in the upper half space model, with unit speed parametrization, is (Q, c)-quasigeodesic.*

Proof. Set $z_0 = x + hi$. By the isometry $z \to \frac{1}{h}(z - z_0)$ of the upper half space, every absolute value path is isometric to a subpath of the path determined by $1 - |t|$ for $|t| < 1$. So it suffices to show that the path determined by $1 - |t|$ for $|t| < 1$ is quasigeodesic.

The line $y = x$ in the upper half space is invariant under the isometry $z \mapsto \lambda z$, and so is a constant distance c_1 from the vertical geodesic given by the vertical y-axis. Nearest point projection from $y = x$ to the vertical axis contracts distances by a constant factor Q_1, so $y = x$ is a (Q_1, c_1)-quasigeodesic.

The absolute value path $1 - |t|$ is therefore a union of two quasigeodesic paths with distinct endpoints, and so is (Q_2, c_2)-quasigeodesic, where the constants depends on Q_1 and c_1, and the Gromov product of the limit points of the two paths based at their common point. □

Definition 140. Suppose that $\alpha(t) = (t, a(t))$ is the path determined by the function $a\colon I \to \mathbb{R}_+$. Given a constant $K \geqslant 0$, the *vertical K-neighborhood of* α, which we shall denote $V_K(\alpha)$, consists of all points whose vertical distance to α (along lines with fixed real part in the upper half space model) in the Euclidean metric is at most K.

We now show that 1-Lipschitz functions contained in bounded vertical neighborhoods of absolute value paths are quasigeodesic.

Proposition 141. *Given $K > 0$ there are constants $Q > 0$ and $c \geqslant 0$ such that for any 1-Lipschitz function $b\colon I \to \mathbb{R}_+$ for which*

- *$b(t) \geqslant 1$, and*
- *the path $\beta(t) = (t, b(t))$ determined by b is contained in a vertical K-neighborhood of an absolute value path,*

the unit speed parametrization of β is (Q, c)-quasigeodesic.

Proof. Let $\alpha(t)$ be a unit speed parametrization of the absolute value path, and let $\beta(t)$ be a unit speed parametrization of β. As β is a unit speed parametrization, $d_{\mathbb{H}^2}(\beta(s), \beta(t)) \leqslant t - s$. We now find a lower bound on distances along the path β.

Let $\alpha(s')$ be the vertical projection of $\beta(s)$, and let $\alpha(t')$ be the vertical projection of $\beta(t)$. The vertical projection from β to α changes distances by at most a factor of e^K, so

$$e^{-K}\text{length}(\alpha([s', t'])) - 2K \leqslant \text{length}(\beta([s, t])) \leqslant e^K\text{length}(\alpha([s', t'])) + 2K. \tag{35}$$

As both α and β have unit speed parametrizations, this shows

$$e^{-K}(t' - s') - 2K \leqslant t - s \leqslant e^K(t' - s') + 2K.$$

As β lies in a vertical K-neighborhood of α,

$$d_{\mathbb{H}^2}(\beta(s), \beta(t)) \geqslant d_{\mathbb{H}^2}(\alpha(s'), \alpha(t')) - 2K.$$

By Proposition 139, the path α is (Q, c)-quasigeodesic, so

$$d_{\mathbb{H}^2}(\beta(s), \beta(t)) \geqslant \tfrac{1}{Q}(t' - s') - c - 2K.$$

Using (35) gives

$$d_{\mathbb{H}^2}(\beta(s), \beta(t)) \geqslant \tfrac{1}{Q}(\tfrac{1}{e^K}(t - s) - 2K) - c - 2K,$$

so β is (Q, c)-quasigeodesic, as required. □

6.4.4 Long segments in complementary regions

We now complete the proof of Lemma 101, showing that segments of the test path over straight segments are quasigeodesic. Recall that there are exactly two types of straight intervals. Either a straight interval is an intersection interval, with both endpoints in the cusps of the ideal complementary region, or the straight interval is the union of two intersection intervals, for ideal complementary regions intersecting in a non-rectangular polygon.

We shall start by showing the result for intersection intervals for a single ideal complementary region.

Lemma 142. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants Q and c such that for any non-exceptional geodesic γ, and any ideal complementary region R with boundary in either Λ_+ or Λ_-, the test path over the intersection interval $\tau_\gamma(I_R)$ is (Q, c)-quasigeodesic.*

Once we have shown this, it will be simple to show that the test path over the union of two intersection segments meeting in a non-rectangular polygon is also quasigeodesic.

Proposition 143. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants $Q \geqslant 1$ and $c > 0$ such that for any non-exceptional geodesic γ intersecting an innermost polygon $P = R_+ \cap R_-$, the test path over the union of the two intersection intervals $\tau_\gamma(I_{R_+} \cup I_{R_-})$ is (Q, c)-quasigeodesic.*

Lemma 101 is then an immediate consequence of Lemma 142 and Proposition 143.

We prove Lemma 142 first. By Proposition 135, if $\gamma(I_R)$ has bounded length then $\tau_\gamma(I_R)$ also has bounded length. So it suffices to show that test paths over sufficiently long intersection intervals are quasigeodesic.

Lemma 144. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants $L_R > 0, Q > 0$ and $c \geqslant 0$ such that for any ideal complementary region R of an invariant lamination, and any non-exceptional geodesic γ that intersects R in a segment of length at least L_R, the test path restricted to the intersection interval I_R is (Q, c)-quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$.*

The R subscript for L_R is to distinguish the constant in Lemma 144 from similar constants in other parts of the paper. Since there are finitely many complementary regions, the order of quantifiers in Lemma 144 is correct, and L_R depends only on (S_h, Λ). We shall choose L_R to be the constant L from Proposition 136, which implies that the sign of the height function does not change on I_R.

An ideal complementary region contains finitely many boundary leaves and extended leaves. We will show that the height function for points of $\gamma(I_R)$ in an ideal complementary region R depends only on the distances to these leaves. To be precise, we show that up to bounded error, the height function is determined by distance to a single leaf of the extended lamination contained in that region. Breaking symmetry, we give the argument for an ideal complementary region R of Λ_+. The same holds for Λ_- with the sign of the height function reversed.

Definition 145. Suppose that S_h a hyperbolic metric and Λ a suited pair of laminations. Given a constant $K \geqslant 0$, an ideal polygon R in $\widetilde{S}_h \setminus \Lambda_+$, a non-exceptional geodesic γ intersecting R, we say that a leaf ℓ of the extended lamination $\overline{\Lambda}_+$ is K-*dominant* on the intersection interval I_R if

- ℓ is contained in R, and
- up to an additive error of K, the radius function $\rho_\gamma(t)$ on the intersection interval I_R equals the radius function $\rho_{\gamma,\ell}(t)$ for the leaf ℓ.

Proposition 146. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants $L > 0, K > 0$ such that for any ideal polygon R in $\widetilde{S}_h \setminus \Lambda_+$ and any non-exceptional geodesic γ intersecting R in a segment of length at least L, there is a leaf ℓ of the extended lamination $\overline{\Lambda}_+$ such that ℓ is K-dominant on the intersection interval I_R.*

We first show that Lemma 144 follows from Proposition 146.

Proof. Suppose that γ intersects an ideal polygon R in $\widetilde{S}_h \setminus \Lambda_+$. By Proposition 146, there is a leaf ℓ of the extended lamination $\overline{\Lambda}_+$ such that ℓ is K-dominant on I_R. As $\gamma(I_R)$ lies in a complementary region, the pseudo-metric distance from γ to ℓ is zero.

As $\ell \in \overline{\Lambda}_+$, the radius function for ℓ is

$$\rho_{\gamma,\ell}(t) = \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(t), \ell^1)}.$$

Let $\gamma(t_\ell)$ be the closest point on γ to ℓ, and assume that the closest distance in $\mathrm{PSL}(2, \mathbb{R})$ is θ_ℓ. Since the height function vanishes if the angle is greater than θ_Λ and since we have chosen $\theta_f \leqslant \theta_0$ in Definition 109, Proposition 69 applies and there is a constant K such that

$$\log \tfrac{1}{\theta_\ell} - |t - t_\ell| - K \leqslant \rho_{\gamma,\ell}(t) \leqslant \log \tfrac{1}{\theta_\ell} - |t - t_\ell| + K.$$

Thus the radius function lives in a vertical K-neighborhood of an absolute value function, and hence the test path is (Q, c)-quasigeodesic by Proposition 141, where Q and c depend only on K, as required. □

We now show that if the projection intervals of two leaves to γ are coarsely nested then the radius function of the leaf with the longer interval is a coarse upper bound for the radius function of the other leaf.

Proposition 147. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Given a constant $K > 0$ there is a constant $L > 0$ such that for any three geodesics γ, ℓ_1 and ℓ_2 with distinct endpoints, such that the projection intervals I_{ℓ_1} and I_{ℓ_2} for ℓ_1 and ℓ_2 onto γ are coarsely nested, i.e. $I_{\ell_2} \subseteq N_K(I_{\ell_1})$, then the radius function ρ_{γ,ℓ_1} determined by ℓ_1 is a coarse upper bound for the radius function ρ_{γ,ℓ_2} determined by ℓ_2, i.e. for all t,*

$$\rho_{\gamma,\ell_2}(t) \leqslant \rho_{\gamma,\ell_1}(t) + L.$$

Proof. Suppose that the smallest $\mathrm{PSL}(2,\mathbb{R})$-distance between γ and ℓ_1 is $\theta_1 > 0$, and assume that this occurs at time t_1, the midpoint of I_{ℓ_1}. Similarly, suppose that the smallest distance between γ and ℓ_2 is θ_2, and assume that this occurs at t_2, the midpoint of I_{ℓ_2}.

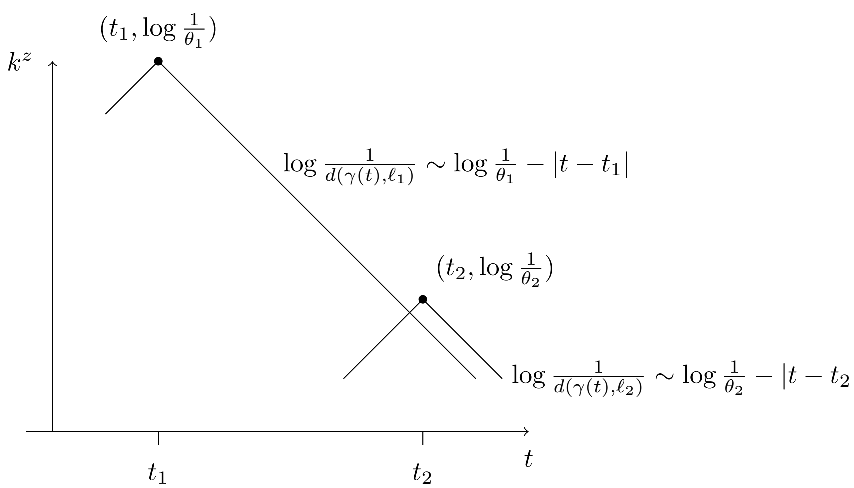

Figure 20: The radius functions for truncated projection intervals with close endpoints.

Recall that the exponential interval $E_{\ell_i} = [t_i - \log \frac{1}{\theta_i}, t_i + \log \frac{1}{\theta_i}]$ is contained in the projection interval I_{ℓ_i}, and furthermore, there is a constant K_1 such that $I_{\ell_i} \subseteq N_{K_1}(E_{\ell_i})$, where K_1 is the constant T_0 from Proposition 25. So if the projection intervals are coarsely nested, $I_{\ell_2} \subseteq N_K(I_{\ell_1})$, then the exponential intervals are also coarsely nested, $E_{\ell_2} \subseteq N_{K+K_1}(E_{\ell_1})$.

From (15), the exponential height function for a leaf is equal to an absolute value function, up to additive error K_2. Thus, for $i = 1, 2$, we have for $t \in E_{\ell_i}$,

$$\log \tfrac{1}{\theta_i} - |t - t_i| - K_2 \leqslant \rho_{\gamma,\ell_i}(t) \leqslant \log \tfrac{1}{\theta_i} - |t - t_i| + K_2,$$

and for $t \notin E_{\ell_i}$, the corresponding radius function is bounded $\rho_{\gamma,\ell_i}(t) \leqslant K_2$.

If $t \notin E_{\ell_1}$, then t is within distance $K + K_1$ of an endpoint of E_{ℓ_2}, so $\rho_{\gamma,\ell_2}(t) \leqslant K + K_1 + K_2$. As $\rho_{\gamma,\ell_1}(t) > 0$ for all t, this implies $\rho_{\gamma,\ell_2}(t) \leqslant \rho_{\gamma,\ell_1}(t) + K + K_1 + K_2$.

We now consider points $t \in E_{\ell_1}$. Consider two absolute value functions $|\cdot|_{I_a} : I_a \to \mathbb{R}$, with $I_a = [t_a - a, t_a + a]$ and $|t|_{I_a} = a - |t_a - t|$ and $|\cdot|_{I_b} : I_b \to \mathbb{R}$, with $I_b = [t_b - b, t_b + b]$ and $|t|_{I_b} = b - |t_b - t|$. If the first interval is contained in the second, $I_a \subseteq I_b$, then $|t|_{I_a} \leqslant |t|_{I_b}$. If the first interval is coarsely contained in the second, $I_a \subseteq N_{K+K_1}(I_b)$, then $|t|_{I_a} \leqslant |t|_{I_b} + K + K_1$. It immediately follows that

$$\rho_{\gamma,\ell_2}(t) \leqslant \rho_{\gamma,\ell_1}(t) + K + K_1 + K_2.$$

The result then follows choosing $L = K + K_1 + K_2$, which only depends on L, and constants depending on the geometry of $\mathrm{PSL}(2, \mathbb{R})$, as required. $\square$

We now show that if I_R is sufficiently long, then the radius function along $\gamma(I_R)$ is equal to the radius function with respect to a single extended leaf in R, up to bounded additive error.

Proposition 148. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. Then there are constants $L_R > 0$ (the constant from Proposition 136) and $K \geqslant 0$ such that for any ideal polygon R in $\widetilde{S}_h \setminus \Lambda_+$, and any non-exceptional geodesic γ such that the intersection interval I_R has length at least L_R, there is a single leaf ℓ of the extended lamination $\overline{\Lambda}_+$ contained in R, such that for all $t \in I_R$*

$$|\rho_\gamma(t) - \rho_{\gamma,\ell}(t)| \leqslant K.$$

Proof. Up to swapping laminations, we may assume that R is an ideal complementary region of Λ_+.

Suppose that γ is a non-exceptional geodesic that intersects R and the length of $\gamma(I_R)$ is at least L_R. By Proposition 136, $h_\gamma(t)$ is always non-negative along I_R. By definition of the height function, $\rho_{\gamma,\overline{\Lambda}_-}(t) \leqslant 1 + \log \frac{1}{\theta_\Lambda}$ for all $t \in I_R$, and so up to bounded additive error, $\rho_\gamma(t) = \rho_{\gamma,\overline{\Lambda}_+}(t)$ for all $t \in I_R$.

We will now identify which leaf ℓ of the extended lamination is K-dominant. For any geodesic ℓ in $\mathbb{H}^2$, the nearest point projection of ℓ to γ equals the segment of γ bounded by the projections of the ideal points of ℓ. Denote by p_i the finitely many ideal points of R, and by z_i their closest point projections on γ. Suppose that z_i and z_j are the outermost points, that is, all other z_u are contained in the segment $[z_i, z_j]$ of γ. We set ℓ to be the leaf of the extended lamination $\overline{\Lambda}_+$ connecting p_i to p_j, which may be a boundary leaf of R.

We now show that for any point t in I_R, the height function $h_\gamma(t)$ equals up to a bounded additive error, the height function $h_{\gamma,\ell}(t)$ determined by this leaf.

Let ℓ_1 and ℓ_2 be the first and the last boundary leaves of R that γ intersects. We have illustrated this in Figure 21 in the case where the leaves ℓ, ℓ_1 and ℓ_2 are all distinct. It may be that one of ℓ_1 or ℓ_2 equals ℓ, but this makes no difference to the argument.

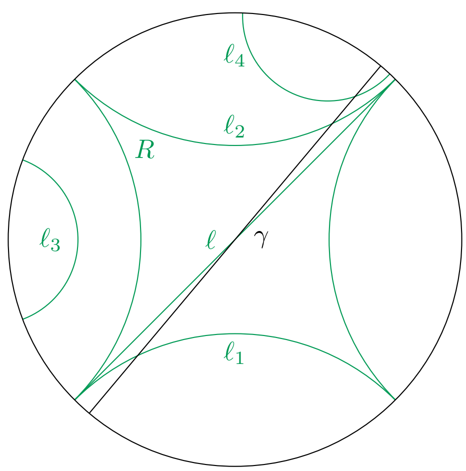

Figure 21: A non-exceptional geodesic intersecting an ideal polygon.

First consider a leaf ℓ' of the extended lamination which is either a boundary leaf of R, or contained in R. By our choice of ℓ, the nearest point projection of ℓ' to γ is contained in the nearest point projection of ℓ to γ. By Proposition 147,

$$\rho_{\gamma,\ell'}(t) \leqslant \rho_{\gamma,\ell}(t) + L,$$

where L is the constant from Proposition 147 with $K = 0$. Note that this same argument works for any leaf (such as ℓ_3 in Figure 21) of the extended lamination $\overline{\Lambda}_+$ contained in a component of $\widetilde{S}_h \setminus R$ disjoint from γ.

Finally, suppose ℓ_4 is a leaf of $\overline{\Lambda}_+$ in one of the two components of $\widetilde{S}_h \setminus R$ that γ intersects. The leaf ℓ_4 itself may or may not intersect γ. Breaking symmetry, suppose that ℓ_4 lies in the component of $\widetilde{S}_h \setminus R$ with boundary ℓ_2, as in Figure 21 above. The same argument holds for the component with boundary ℓ_1.

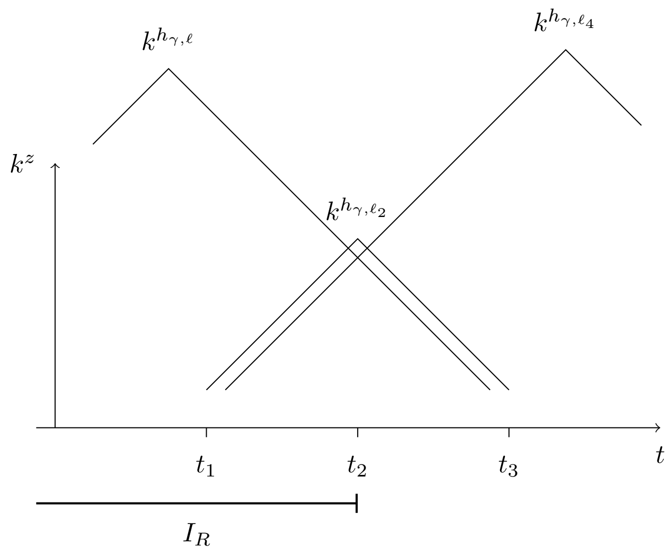

Figure 22: Radius functions for leaves with overlapping projection intervals.

Let $\gamma(t_2)$ be the intersection point of γ with the boundary leaf ℓ_2. Note that

- $\gamma(t_2)$ is the endpoint of the intersection interval I_R, and
- $\gamma(t_2)$ is the midpoint of the nearest point projection interval I_{ℓ_2} of ℓ_2 onto γ.

Let $\gamma(t_1)$ and $\gamma(t_3)$ be the initial and terminal points of I_{ℓ_2}. Let I_ℓ and I_{ℓ_4} be the nearest point projection intervals onto γ for the leaves ℓ and ℓ_4. Then $I_{\ell_2} \subseteq I_\ell$, with common endpoint $\gamma(t_3)$, and $I_{\ell_4} \subseteq I_{\ell_2}$, with common initial point $\gamma(t_1)$. This is illustrated in Figure 22.

As $I_{\ell_2} \subseteq I_\ell$, by Proposition 147 the radius function $\rho_{\gamma,\ell}$ is a coarse upper bound for the radius function ρ_{γ,ℓ_2}. As the initial point of I_{ℓ_4} is further along γ than $\gamma(t_1)$, the initial point of I_{ℓ_2}, the radius function ρ_{γ,ℓ_2} is a coarse upper bound for ρ_{γ,ℓ_4} on $[t_1, t_2]$, the first half of $I_{\ell_2} = [t_1, t_3]$ on which both ρ_{γ,ℓ_2} and ρ_{γ,ℓ_4} are increasing. As t_2 is the endpoint of $\gamma(I_R)$, we deduce that $\rho_{\gamma,\ell}$ is a coarse upper bound on I_R for the radius functions corresponding to all leaves of the extended lamination $\overline{\Lambda}_+$, as required. □

Proposition 146 now follows directly from Proposition 148.

Finally, we prove Proposition 143, that the test path over the union of two intersections intervals meeting in a non-rectangular polygon is also quasigeodesic.

Proof. Proof (of Proposition 143) Let p_1 be the endpoint of I_R in the innermost polygon P. Let p_0 be the other endpoint of I_R, and let p_2 be the endpoint of $I_{R'}$ disjoint from I_R. As both $\tau_\gamma(I_R)$ and $\tau_\gamma(I_{R'} \setminus I_R)$ are quasigeodesic, it suffices to prove that the Gromov product $(p_0, p_2)_{p_1}$ is bounded.

By Proposition 146, there is an extended leaf ℓ_+ in R_+ that is K-dominant for $\gamma(I_{R_+})$. In particular, the test path over $\gamma_{I_{R_+}}$ lies in a bounded neighborhood of a half space in the ladder $F(\ell_+)$. Similarly, the complementary region of Λ_- containing the cusp C has a segment of an extended leaf ℓ_- that is K-dominant for $\gamma(I_C)$. Thus the test path over $\gamma(I_R)$ lies in a bounded neighborhood of a half space in the ladder $F(\ell_-)$. Let q be the intersection point of the two extended leaves ℓ_+ and ℓ_-. Both p and q_1 lie in P and so are a bounded distance apart. The union of the two quasigeodesics is thus contained in a bounded neighborhood of the union of two half spaces meeting along the suspension flow line through q. Each half space is quasi-isometric to a half space in $\mathbb{H}^2$ with geodesic boundary, so the union of the two half spaces is quasi-isometric to $\mathbb{H}^2$. The nearest point projection of the first quasigeodesic to suspension flow line $F(q)$ is contained in a bounded neighborhood of the positive half of $F(q)$, and the nearest point projection of the second quasigeodesic is contained in a bounded neighborhood of the negative part of $F(q)$. Therefore the Gromov product of the endpoints is bounded, and so the union of the two quasigeodesics is a quasigeodesic. □

6.5 Vertical projection to the test path is distance decreasing

Finally, we combine the fact that the test path is quasigeodesic with the fact that the height function is Lipschitz, to show that the vertical projection from $\iota(\gamma)$ to τ_γ is coarsely distance decreasing.

Proposition 56. Suppose that (S_h, Λ) is a hyperbolic metric on S together with a suited pair of measured laminations. There are constants K and c such that for any non-exceptional geodesic γ with unit speed parametrization, and any s and t,

$$d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(s), \tau_\gamma(t)) \leqslant K d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(s)), \iota(\gamma(t))) + c,$$

where here the test path has the parametrization inherited from the unit speed parametrization on γ.

We remark that we do not show that the nearest point projection of $\iota(\gamma(t))$ to the geodesic $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$ is close to the corresponding test path point $\tau_\gamma(t)$. This is equivalent to the statement that there is an upper bound on the length of the fellow traveling interval between the vertical flow lines from $\iota(\gamma(t))$ to $\tau_\gamma(t)$ and the geodesic $\overline{\gamma}$. Although this may be true for our choice of test path, this property does not hold for all quasigeodesic paths with the same endpoints as $\overline{\gamma}$, and so depends on the exact choice of quasigeodesic.

Proof. Let γ be a non-exceptional unit speed geodesic in $\widetilde{S}_h$, and let $\tau_\gamma(t)$ be the test path with the (non-unit speed) parametrization determined by $\gamma(t)$. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ.

Let $k > 1$ be the constant from the definition of the Cannon-Thurston metric, and let δ_3 be the constant of hyperbolicity for $\widetilde{S}_h \times \mathbb{R}$. We will choose $K = 1 + 2/\log k$, and $c = 12\delta_3 + 6L$. Here L is the Morse constant such that the test path τ_γ is contained in an L-neighborhood of the geodesic $\overline{\gamma}$. For notational convenience, set $D = d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_1)), \iota(\gamma(t_2)))$.

Let p_i be the nearest point projection of the test path location $\tau_\gamma(t_i)$ to $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, and let q_i be the nearest point projection of $\iota(\gamma(t_i))$ to $\overline{\gamma}$.

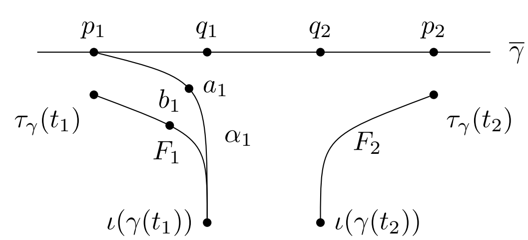

Figure 23: The geodesic $\overline{\gamma}$ and the vertical flow lines.

We denote by F_i the vertical flow segment from $F_0(\iota(\gamma(t_i)))$ to $\tau_\gamma(t_i) = F_{h_{\gamma(t_i)}}(\iota(\gamma(t_i)))$. The path α_i consisting of the union of the two geodesics $[p_i, q_i] \cup [q_i, \iota(\gamma(t_i))]$ is close to being a geodesic. In particular, by [KS24, Lemma 1.102] there is a point a_i on α_i distance at most $2\delta_3$ from q_i. By thin triangles and the distance from $\tau_\gamma(t_i)$ to p_i being at most L, there is a point b_i on F_i within distance $L + 3\delta_3$ of q_i.

Suppose that q_1 lies between p_1 and p_2. As τ_γ is contained in an L-neighborhood of $\overline{\gamma}$, there is a point $\tau_\gamma(t)$ distance at most L from q_1, with $t_1 \leqslant t \leqslant t_2$. As the height function is $(1/\log k)$-Lipschitz, the height difference between $\tau_\gamma(t_1)$ and $\tau_\gamma(t)$ is at most $D/\log k$, and so the height difference between p_1 and q_1 is at most $D/\log k + 2L$.

By the same argument from the previous two paragraphs applied to F_2, if q_2 lies between p_1 and p_2, then the distance between p_2 and q_2 is also at most $D/\log k + 2L$.

As q_1 is the nearest point projection of $\iota(\gamma(t_1))$ to $\overline{\gamma}$, the path consisting of the concatenation of the three geodesics $[\iota(\gamma(t_1)), q_1] \cup [q_1, q_2] \cup [q_2, \iota(\gamma(t_2))]$ is close to being a geodesic. By [KS24, Lemma 1.120], if the distance between q_1 and q_2 is at least $8\delta_3$, then the distance between $\iota(\gamma(t_1))$ and $\iota(\gamma(t_2))$ is at least $|[q_1, q_2]| - 12\delta_3$.

The distance in $\widetilde{S}_h \times \mathbb{R}$ between $\iota(\gamma(t_n))$ and $\iota(\gamma(t_{n+1}))$ is at most D, so the distance between q_1 and q_2 is at most $D + 12\delta_3$. This implies that the distance between p_1 and p_2 is at most $D + 12\delta_3 + 2D/\log k + 4L$, and so the distance between $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ is at most $(1 + 2/\log k)D + 12\delta_3 + 6L$, as required. □

Appendix A Distance bounds in $\mathbb{H}^2$

In this section we record some standard estimates on distances between geodesics in the hyperbolic plane $\mathbb{H}^2$. We start by finding bounds on the distances in $\mathbb{H}^2$ between two non-intersecting geodesics distance θ apart.

Proposition 149. *Suppose that $\theta \leqslant 1$ is a positive constant and suppose that γ_1 and γ_2 are two bi-infinite geodesics in $\mathbb{H}^2$ which do not intersect and are distance θ apart. Suppose that γ_1 is parametrized with unit speed such that $\gamma_1(0)$ is the closest point on γ_1 to γ_2. Then*

$$\tfrac{1}{3}\theta e^{|t|} \leqslant d_{\mathbb{H}^2}(\gamma_1(t), \gamma_2) \leqslant \tfrac{3}{2}\theta e^{|t|}, \text{ if } |t| \leqslant \log \tfrac{1}{\theta}.$$

Furthermore, the lower bound at $|t| = \log \frac{1}{\theta}$ holds for all $|t| \geqslant \log \frac{1}{\theta}$, i.e. $d_{\mathbb{H}^2}(\gamma_1(t), \gamma_2) \geqslant \frac{1}{3}$.

Proof. Choose unit speed parametrizations of γ_1 and γ_2 so that their closest points are $\gamma_1(0)$ and $\gamma_2(0)$. Let p be the closest point on γ_2 to $\gamma_1(t)$, and set $d = d_{\mathbb{H}^2}(\gamma_1(t), \gamma_2) = d_{\mathbb{H}^2}(\gamma_1(t), p)$. As the distances are symmetric for t and $-t$, it suffices to consider the case $t \geqslant 0$.

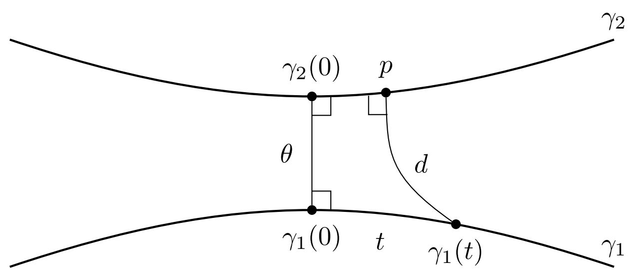

Figure 24: The geodesic γ_3 connecting non-adjacent limit points of γ_1 and γ_2.

The four points $\gamma_1(0), \gamma_2(0), \gamma_1(t)$ and p determine a hyperbolic quadrilateral with three right angles, which is known is as a Lambert quadrilateral, and its sides satisfy

$$\sinh d = \cosh t \sinh \theta, \tag{36}$$

see for example [Mar96, Section 32.2]. We will use the following elementary estimates: $x \leqslant \sinh x \leqslant \frac{3}{2}x$ for $0 \leqslant x \leqslant \sinh(1) < 1.18$, and $\frac{1}{2}e^x \leqslant \cosh x \leqslant e^x$ for all values of x. Applying these estimates for $0 \leqslant \theta \leqslant 1$, and $0 \leqslant d \leqslant \sinh(1)$ gives

$$\tfrac{1}{3}\theta e^t \leqslant d \leqslant \tfrac{3}{2}\theta e^t,$$

for $|t| \leqslant \log \frac{1}{\theta}$, as for these values of t, $d \leqslant \sinh(1)$. Differentiating (36) shows that for fixed θ, d is increasing in t, and so for all $t \geqslant \log \frac{1}{\theta}$, the lower bound at $t = \log \frac{1}{\theta}$ holds, i.e. $d_{\mathbb{H}^2}(\gamma_1(t), \gamma_2) \geqslant \frac{1}{3}$, as required. □

We now find bounds on the distances between two geodesics in $\mathbb{H}^2$ which intersect at angle θ.

Proposition 150. *Suppose that γ_1 and γ_2 are geodesics in $\mathbb{H}^2$ which intersect at angle $\theta < 1$, and suppose that γ_1 has unit speed parametrization so that the intersection point is $\gamma_1(0)$. Then*

$$\tfrac{1}{8}\theta(e^t - 1) \leqslant d_{\mathbb{H}^2}(\gamma_1(t), \gamma_2) \leqslant \tfrac{1}{2}\theta e^t, \text{ if } |t| \leqslant \log \tfrac{1}{\theta},$$

and furthermore, the lower bound at $t = \log \frac{1}{\theta}$ holds for all $t \geqslant \log \frac{1}{\theta}$, i.e. $d_{\mathbb{H}^2}(\gamma_1(t), \gamma_2) \geqslant \frac{1}{8}(1 - \theta)$.

Proof. Let $\gamma_1(t)$ be the point distance t along γ_1 from the intersection point, and let d be the distance from $\gamma_1(t)$ to γ_2.

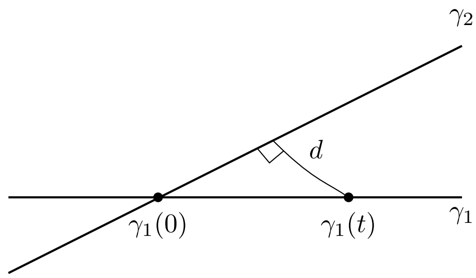

Figure 25: The geodesics of γ_1 and γ_2 intersect at angle θ.

Using the sine formula for right angled triangles in hyperbolic space gives

$$\sin\theta = \frac{\sinh d}{\sinh t}.$$

Using the elementary estimates: $\frac{1}{2}\theta \leqslant \sin\theta \leqslant \theta$ for $0 \leqslant \theta \leqslant 1$ and $\frac{1}{2}(e^t - 1) \leqslant \sinh t \leqslant \frac{1}{2}e^t$ for $t \geqslant 0$, we get

$$\tfrac{1}{4}\theta(e^t - 1) \leqslant \sinh d \leqslant \tfrac{1}{2}\theta e^t$$

For $t \leqslant \log\frac{1}{\theta}$, $\sinh d \leqslant \frac{1}{2} < 1$, so have the elementary estimate $d \leqslant \sinh d \leqslant 2d$ for $0 \leqslant d \leqslant 1$, which gives

$$\tfrac{1}{8}\theta(e^t - 1) \leqslant d \leqslant \tfrac{1}{2}\theta e^t$$

as required.

Finally, the bound $\sinh(d) \geqslant \frac{1}{4}(e^t - 1)$ holds for all $t \geqslant 0$, and e^t is increasing, so for all $t \geqslant \log\frac{1}{\theta}$, $\sinh d \geqslant \frac{1}{4}(1-\theta)$. The right hand side takes values in $[0, \frac{1}{4}]$, so d takes values in $[0, \sinh^{-1}(\frac{1}{4}) < 1]$, and we may apply the elementary bound for $\sinh d$, so $d \geqslant \frac{1}{8}(1-\theta)$ for all $t \geqslant \log\frac{1}{\theta}$. □

Finally, we prove the bounds on the size of nearest point projection intervals.

Proposition 25. There is a constant $T_0 > 0$ such that for any geodesics γ_1 and γ_2 in $\mathbb{H}^2$ that

- intersect at an angle $0 < \theta \leqslant \pi/2$, and
- γ_1 is parametrized with unit speed so that $\gamma_1(0)$ is the point of intersection,

then the image of γ_2 under nearest point projection to γ_1 equals $\gamma_1([-T, T])$, where

$$\log\tfrac{1}{\theta} \leqslant T \leqslant \log\tfrac{1}{\theta} + T_0.$$

Proof. We may parametrize γ_1 and γ_2 with unit speed so that they intersect at $\gamma_1(0) = \gamma_2(0)$. Let $\gamma_1(s)$ be the nearest point projection of $\gamma_2(t)$ to γ_1. Up to reversing the parametrization for γ_1 we may assume that $s \geqslant 0$ whenever $t \geqslant 0$. Then the three points $\gamma_1(0), \gamma_2(t)$ and $\gamma_1(s)$ form a right angled triangle, with right angle at $\gamma_1(s)$ and hypotenuse of length t. This is illustrated in Figure 26.

For right angled triangles in $\mathbb{H}^2$, we have

$$\cos\theta = \frac{\tanh s}{\tanh t}.$$

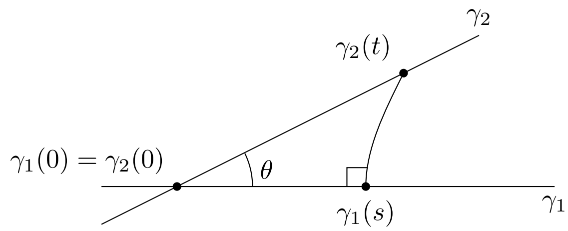

Figure 26: The nearest point projection from one leaf to an intersecting leaf.

We will take the limit as t tends to infinity, so we may replace $\tanh(t)$ by one. We will use the elementary bounds that for $0 \leqslant x \leqslant \pi/2$, $1 - x^2/2 \leqslant \cos x \leqslant 1 - x^2/4$. This gives

$$1 - \tfrac{1}{2}\theta^2 \leqslant \tanh s \leqslant 1 - \tfrac{1}{4}\theta^2.$$

Using the formula for $\tanh^{-1}(x) = \frac{1}{2}\log\frac{1+x}{1-x}$, we get

$$\tfrac{1}{2}\log\frac{2 - \frac{1}{2}\theta^2}{\frac{1}{2}\theta^2} \leqslant s \leqslant \tfrac{1}{2}\log\frac{2 - \frac{1}{4}\theta^2}{\frac{1}{4}\theta^2}.$$

$$\tfrac{1}{2}\log\frac{4}{\theta^2} - 1 \leqslant s \leqslant \tfrac{1}{2}\log\frac{2}{\frac{1}{4}\theta^2}.$$

$$\tfrac{1}{2}\log\frac{3}{\theta^2} \leqslant s \leqslant \tfrac{1}{2}\log\frac{8}{\theta^2}.$$

$$\log\tfrac{1}{\theta} + \tfrac{1}{2}\log 3 \leqslant s \leqslant \log\tfrac{1}{\theta} + \tfrac{1}{2}\log 8.$$

As the constant on the left is positive, we can choose $T_0 = \frac{1}{2}\log 8 \leqslant 2$. □

Appendix B Distance bounds in PSL$(2, \mathbb{R})$

The fellow traveling results in this section are standard, but we give detailed proofs for the convenience of the reader. All constants in this section depend only on the geometry of PSL$(2, \mathbb{R})$, and in particular do not depend on a choice of hyperbolic surface S_h or the suited pair of laminations Λ.

We abuse notation by using the same notation for geodesics in $\mathbb{H}^2$ and their lifts in PSL$(2, \mathbb{R})$.

Proposition 69. There are constants $\theta_0 > 0$ and $L_0 \geqslant 1$ such that for any geodesics γ_1 and γ_2 in $\mathbb{H}^2$

- whose lifts in PSL$(2, \mathbb{R})$ are distance $\theta \leqslant \theta_0$ apart, and
- the lift γ_1 has unit speed parametrization such that $\gamma_1(0)$ is the closest point to γ_2,

then for any t such that $|t| \leqslant \log\frac{1}{\theta}$

$$\tfrac{1}{L_0}\theta e^{|t|} \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant L_0\theta e^{|t|}.$$

Furthermore, the lower bound at $|t| = \log\frac{1}{\theta}$ holds for all $|t| \geqslant \log\frac{1}{\theta}$.

For t reasonably large, distances in $\mathbb{H}^2$ give reasonable bounds on distances in PSL$(2, \mathbb{R})$. However, they are not precise enough for our purposes close to $t = 0$. For small t we use the fact that the left invariant metric is bilipschitz to the corresponding matrix norm on PSL$(2, \mathbb{R})$.

Proposition 151. *There are constants $\theta_0 > 0$, $L \geqslant 1$ and $K \leqslant \log \frac{L}{\theta_0}$, such that for any two geodesics γ_1 and γ_2 in $\mathbb{H}^2$*

- *whose lifts in $PSL(2, \mathbb{R})$ are distance $\theta \leqslant \theta_0$ apart, and*
- *the lift $\gamma_1(0)$ has unit speed parametrization such that $\gamma_1(0)$ is the closest point to γ_2,*

then for all $|t| \leqslant \log \frac{1}{\theta} - K$,

$$\tfrac{1}{L}\theta e^{|t|} \leqslant d_{PSL(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant L\theta e^{|t|}.$$

In particular, $\gamma_1(t)$ lies within a θ_0-neighborhood of γ_2 for a symmetric interval of length at least $2(\log \frac{1}{\theta} - K)$ centered at $t = 0$.

We will use the fact that there is a neighborhood U of the identity matrix in $\mathrm{PSL}(2, \mathbb{R})$ such that any left invariant metric is bilipschitz to the metric arising from any matrix norm, see for example [EW11, Lemma 9.12], where this is shown for any Lie group. In fact, this holds as long as U is the image of a bounded neighborhood V of the zero matrix in the Lie algebra $\mathrm{sl}(2, \mathbb{R})$ on which the exponential map is injective.

Proposition 152 ([EW11, Lemma 9.12])**.** *There is a constant $\theta_0 > 0$ such that for any matrix norm $\|A\|$, and any left invariant metric $d_{PSL(2,\mathbb{R})}(A, B)$ on $PSL(2, \mathbb{R})$, there is a constant $L_0 > 0$ such that for any matrix $A \in PSL(2, \mathbb{R})$, with $d_{PSL(2,\mathbb{R})}(I, A) \leqslant \theta_0$,*

$$\tfrac{1}{L_0} \|I - A\| \leqslant d_{PSL(2,\mathbb{R})}(I, A) \leqslant L_0 \|I - A\|.$$

We shall use the matrix norm determined by the Frobenius or Hilbert-Schmidt inner product $\langle A, B\rangle = 2\mathrm{tr}(A^T B)$ on 2×2 matrices. The scaling factor of 2 is chosen so that the following basis for $\mathrm{sl}(2, \mathbb{R})$ is orthonormal,

$$\{A_1, A_2, A_3\} = \left\{ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right\}.$$

We will make use of the following standard forms for a pair of geodesics in $\mathbb{H}^2$. Let α and β be oriented geodesics. Then there is an oriented geodesic γ whose positive limit point is the same as the positive limit point of α, and whose negative limit point is the same as the negative limit point for β. Up to the action of $\mathrm{PSL}(2, \mathbb{R})$, we may assume that γ is given by

$$\gamma(t) = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}.$$

All geodesics with the same positive limit point are known as the stable manifold for γ and are given by

$$\alpha_s(t) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \gamma(t).$$

Similarly, all geodesics with the same negative limit point as γ, known as the unstable manifold for γ, are given by

$$\beta_{s'}(t) = \begin{bmatrix} 1 & 0 \\ s' & 1 \end{bmatrix} \gamma(t).$$

Given geodesics α and β, we can, by a reflection if required and a translation of the parametrization of γ, arrange that in the standard form $\alpha = \alpha_s$ with $s > 0$ and $\beta = \beta_{s'}$ with $s' = \pm s$.

Proposition 153. *For $a + b \leqslant \theta_0 / L_0 \sqrt{2}$,*

$$\tfrac{\sqrt{2}}{L_0} \sqrt{a^2 + b^2} \leqslant d_{PSL(2,\mathbb{R})} \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \right) \leqslant L_0 \sqrt{2}(a + b)$$

Proof. Note that $\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix} = e^{Ua}$, where $U = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$. In our chosen orthonormal basis for $\mathrm{sl}(2,\mathbb{R})$, the length of the matrix U is $\sqrt{2}$. Since $L_0 \left\|I - e^{Ua}\right\| = L_0 \left\|Ua\right\| = L_0\sqrt{2}a \leqslant \theta_0$, we may use the upper bound in Proposition 152 to obtain that the distance of e^{Ua} from the identity matrix is at most $L_0\sqrt{2}a$. The required upper bound follows directly from the triangle inequality.

For the lower bound, we may again use Proposition 152 to obtain

$$d_{\mathrm{PSL}(2,\mathbb{R})}\left(\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\right) \geqslant \tfrac{1}{L_0}\left\|I - \begin{bmatrix}1 & -a\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\right\| \geqslant \tfrac{1}{L_0}\left\|\begin{bmatrix}ab & a\\ -b & 0\end{bmatrix}\right\|$$
$$\geqslant \tfrac{\sqrt{2}}{L_0}\sqrt{a^2b^2 + a^2 + b^2} \geqslant \tfrac{\sqrt{2}}{L_0}\sqrt{a^2 + b^2},$$

as required. □

Proposition 154. *For s such that $2s \leqslant \theta_0/L_0\sqrt{2}$, the distance θ between the geodesics α_s and β_s satisfies*

$$\tfrac{2}{L_0}s \leqslant \theta \leqslant L_0 2\sqrt{2}s.$$

Proof. Since an upper bound may be obtained by picking any two points, we have

$$d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha_s, \beta_s) \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha_s(0), \beta_s(0)).$$

By the previous proposition, $d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha_s(0), \beta_s(0)) \leqslant L_0 2\sqrt{2}s$.

We now obtain a lower bound. By definition,

$$\theta = \inf_{t,t'} d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha_s(t), \beta_s(t')).$$

and moreover the right hand side will be infimized over small values of t and t'. In particular, we may use the lower bounds from Proposition 152, to obtain

$$\theta \geqslant \inf_{t,t'} \tfrac{1}{L_0}\left\|I - \begin{bmatrix}e^{t/2} & se^{-t/2}\\ 0 & e^{-t/2}\end{bmatrix}^{-1}\begin{bmatrix}e^{t'/2} & 0\\ se^{t'/2} & e^{-t'/2}\end{bmatrix}\right\|.$$

Combining the matrix terms gives

$$\theta \geqslant \tfrac{1}{L_0}\inf_{t,t'}\left\|\begin{bmatrix}1 - (1+s^2)e^{(t'-t)/2} & -se^{-(t+t')/2}\\ -se^{(t+t')/2} & 1 - e^{(t-t')/2}\end{bmatrix}\right\|.$$

The matrix norm is the sum of the squares of the entries. So we may use the sum of the squares of the off-diagonal entries as a lower bound. We obtain

$$\theta \geqslant \tfrac{\sqrt{2}}{L_0}\inf_{t,t'}\sqrt{s^2e^{-(t+t')} + s^2e^{t+t'}}$$

which is minimized when $t + t' = 0$, so

$$\theta \geqslant \tfrac{2}{L_0}s,$$

as required. □

Proposition 155. *There are constants $\theta_0 > 0$ and $L_0 \geqslant 0$ and $K = \log\frac{2L_0}{\theta_0}$, such that for any two distinct geodesics α and β distance $\theta \leqslant \theta_0$ apart, there are unit speed parametrizations $\alpha(t)$ and $\beta(t)$ such that for $|t| \leqslant \log\frac{1}{\theta} - K$,*

$$\tfrac{1}{2L_0^2}\theta e^{|t|} \leqslant d_{PSL(2,\mathbb{R})}(\alpha^1(t), \beta^1(t)) \leqslant \sqrt{2}L_0^2\theta e^{|t|}.$$

Proof. Up to the $\mathrm{PSL}(2,\mathbb{R})$ action, we may assume that α and β have the standard forms $\alpha = \alpha_s(t)$ and $\beta = \beta_s(t)$, defined above. Since the distance is left invariant, we have

$$d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha_s^1(t), \beta_s^1(t)) = d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma^1(-t)\alpha_s^1(t), \gamma^1(-t)\beta_s^1(t))$$

Note that $\gamma^1(-t)\alpha_s^1(t) = \begin{bmatrix} 1 & se^{-t} \\ 0 & 1 \end{bmatrix}$ and $\gamma^1(-t)\beta_s^1(t) = \begin{bmatrix} 1 & 0 \\ \pm se^{t} & 1 \end{bmatrix}$. For the small values of t stated in the hypothesis, we use the bounds from Proposition 153 to obtain

$$\tfrac{\sqrt{2}}{L_0} se^{|t|} \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha^1(t), \beta^1(t)) \leqslant L_0 2\sqrt{2} se^{|t|}$$

By Proposition 154, we obtain from above

$$\tfrac{1}{2L_0^2}\theta e^{|t|} \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\alpha^1(t), \beta^1(t)) \leqslant \sqrt{2}L_0^2 \theta e^{|t|}$$

as required. □

We have obtained bounds on the distance from $\gamma_1^1(t)$ to $\gamma_2^1(t)$. We wish to show that this gives bounds on the distance from $\gamma_1^1(t)$ to γ_2^1. The upper bound is immediate. We start by showing a lower bound that holds in a general geodesic metric space.

Proposition 156. *Suppose that (X, d) is a geodesic metric space. Suppose that γ_1 and γ_2 are geodesics distance θ apart with unit speed parametrizations chosen such that $\gamma_1(0)$ and $\gamma_2(0)$ are closest points between γ_1 and γ_2. Then for all t,*

$$d_X(\gamma_1(t), \gamma_2) \geqslant \tfrac{1}{2}\left(d_X(\gamma_1(t), \gamma_2(t)) - \theta\right).$$

Proof. Suppose that the point of γ_2 closest to $\gamma_1(t)$ is $\gamma_2(r)$, and suppose that it lies distance s from $\gamma_1(t)$. We have illustrated this in Figure 27.

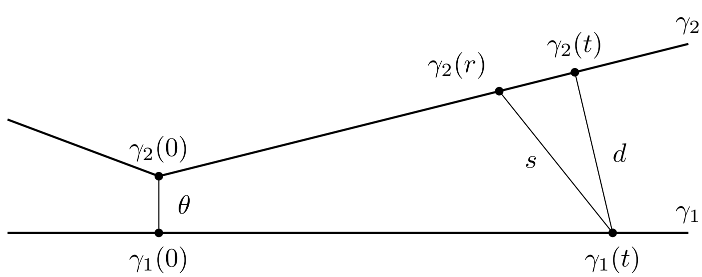

Figure 27: A lower bound for the distance from $\gamma_1(t)$ to γ_2.

Applying the triangle inequality to the path from $\gamma_1(0)$ to $\gamma_1(t)$ via $\gamma_2(0)$ and $\gamma_2(r)$, gives

$$t \leqslant \theta + r + s.$$

Similarly, applying the triangle inequality to the path from $\gamma_1(t)$ to $\gamma_2(t)$ via $\gamma_2(r)$, gives

$$d \leqslant s + t - r.$$

Adding these two inequalities gives $d \leqslant \theta + 2s$, equivalently, $s \geqslant \frac{1}{2}(d - \theta)$, as required. □

We now complete the proof of Proposition 151.

Proof of Proposition 151. By Proposition 155 there is a constant $L \geqslant 1$ such that for any geodesics γ_1 and γ_2, distance $\theta \leqslant \theta_0$ apart, with unit speed parametrizations $\gamma_1(t)$ and $\gamma_2(t)$, such that the distance between $\gamma_1^1(0)$ and $\gamma_2^1(0)$ is equal to θ,

$$\tfrac{1}{L}\theta e^t \leqslant d = d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1(t)) \leqslant L\theta e^t.$$

We now show the following estimate for the distance from $\gamma_1(t)$ to γ_2.

$$\tfrac{1}{4L}\theta e^t \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant L\theta e^t. \tag{37}$$

Since the distance from $\gamma_1^1(t)$ to $\gamma_2^1(t)$ is an upper bound for the distance from $\gamma_1^1(t)$ to γ_2^1, the upper bound follows immediately. For the lower bound, Proposition 156 implies that $d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \geqslant \frac{1}{2}(d - \theta)$. For $d \geqslant 2\theta$, we obtain $d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \geqslant \frac{1}{4}d \geqslant \frac{1}{4L}\theta e^t$. If $d \leqslant 2\theta$ instead, then $t \leqslant \log 8L$. In this case, $\frac{1}{4L}\theta e^t \leqslant \frac{\theta}{2}$. Since $d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \geqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1, \gamma_2^1) = \theta$, the lower bound holds trivially.

The estimate for $d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1)$ in (37) is valid as long as the bilipschitz bounds hold in a small neighborhood of the identity in $\mathrm{PSL}(2,\mathbb{R})$, i.e. as long as $d \leqslant \theta_0$, and this holds as long as $t \leqslant \log\frac{1}{\theta} + \log\frac{\theta_0}{L}$. Therefore we may choose $K = -\log\frac{\theta_0}{L}$, which is non-negative as $\theta_0 \leqslant 1$ and $L \geqslant 1$. Furthermore, this choice of K implies that $\gamma_1^1(t)$ lies within a θ_0-neighborhood of γ_2^1 for an interval of size at least $2(\log\frac{1}{\theta} - K)$, as required. □

Consider the map $p\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2$ given by applying the matrix A to i in the upper half space as a Möbius transformation, i.e.

$$p\colon A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{ai+b}{ci+d}.$$

Recalling our basis for the Lie algebra, note that

$$e^{tA_1} = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix},$$

and so $p(e^{tA_1}) = e^t i$, which is a unit speed geodesic in the upper half space. Similarly $p(e^{tA_2})$ is the unit speed geodesic obtained by rotating the geodesic for e^{tA_1} by $\pi/2$ clockwise, and e^{tA_3} gives a rotation about i. Thus, the derivative at I of the map $p\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2$ is given by the projection matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where we use the basis $\{i, 1\}$ for the tangent space to i in the upper half space. As the map is equivariant, it is distance non-increasing, and so distances in $\mathbb{H}^2$ are lower bounds for distances in $\mathrm{PSL}(2,\mathbb{R})$. Since the kernel of the map is the compact subgroup $SO(2) \cong S^1$, which has diameter $\pi/2$, we get

$$d_{\mathbb{H}^2}(p(A), p(B)) \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(A, B) \leqslant d_{\mathbb{H}^2}(p(A), p(B)) + \pi. \tag{38}$$

We may now complete the proof of Proposition 69 by combining the large t estimates from distances in $\mathbb{H}^2$ with the small t estimates from Proposition 151.

Proof of Proposition 69. For small values of t, we have the bounds from Proposition 151, i.e. there are constants θ_0, L and $K = \log\frac{L}{\theta_0}$ such that for $\theta \leqslant \theta_0$ and $|t| \leqslant \log\frac{1}{\theta} - K$,

$$\tfrac{1}{L}\theta e^t \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant L\theta e^t.$$

We now use the distance estimates in $\mathbb{H}^2$ to extend these bounds to $|t| \leqslant \log\frac{1}{\theta}$. Using (38), and the distance bounds from Proposition 149 and Proposition 150, we obtain the following bounds for distances in $PSL(2,\mathbb{R})$.

$$\tfrac{1}{8}(e^t - 1) \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant \tfrac{3}{2}\theta e^t + \pi.$$

So for $t \in [\log \frac{1}{\theta} - 9, \log \frac{1}{\theta}]$,

$$\frac{1}{M}\theta e^t \leqslant d_{\mathrm{PSL}(2,\mathbb{R})}(\gamma_1^1(t), \gamma_2^1) \leqslant M\theta e^t,$$

for $M = \frac{3}{2} + \pi e^9 \leqslant 10^5$.

We can now choose the constant L_0 in the statement of Proposition 69 to be the maximum of M and L from Proposition 151. □

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