Tropical methods for Teichmüller spaces

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Tropical Geometry

General Idea

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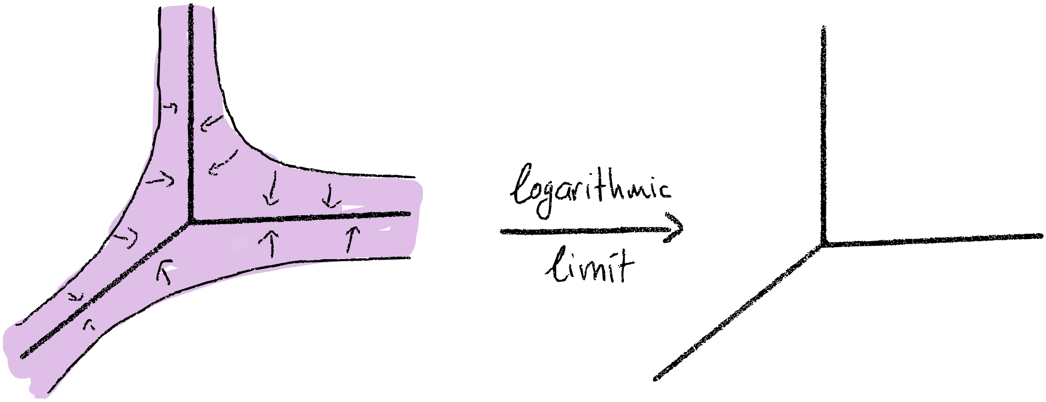
$\rightarrow$ Construct combinatorial analogue trop(X) of X.

$\rightarrow$ Study combinatorial properties of trop(X).

$\rightarrow$ Recover algebraic properties of X from these combinatorial properties.

From curves to limits



Algebraic foundations

Idea: Emulate behavior of exponents under the limit

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From algebraic to tropical

Definition

Let K be a non-archimedean field with valuation. For $f = \sum_{u \in \mathbb{N}^n} c_u x^u \in K[x_1^{\pm}, \ldots, x_n^{\pm}]$ we define the *tropicalization*

$$\mathrm{trop}(f) = \bigoplus_{u \in \mathbb{N}^n} \mathrm{val}(c_u) \odot x^{\odot u} \in \mathbb{T}[x_1, \ldots, x_n].$$

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Example

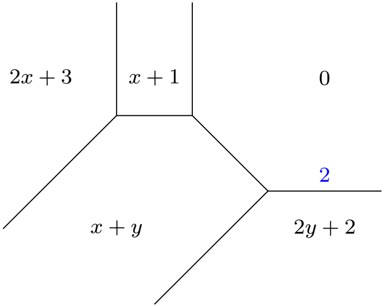
$$\begin{aligned}
&\mathrm{trop}((t^2 + t^5)y^2 + (4 + t^9)xy + (t^3)x^2 + ty + tx + 1) \\
&\quad = 2 \odot y^{\odot 2} \oplus 0 \odot x \odot y \oplus 3 \odot x^{\odot 2} \oplus 1 \odot y \oplus 1 \odot x \oplus 0 \\
&\quad = \min\{2y + 2, x + y, 2x + 3, y + 1, x + 1, 0\}
\end{aligned}$$

Tropical varieties

Definition

Let $f \in \mathbb{T}[x_1, \ldots, x_n]$. The *tropical variety* of f is the set

$$V(f) = \{u | \min \text{ in } f(u) \text{ is attained at least twice}\}$$



Subdivisions

Theorem (Kapranov)

Let $f \in K[x_1^{\pm}, \ldots, x_n^{\pm}]$. Then $-V(\mathrm{trop}(f))$ is dual to the induced subdivision of Newt(f) by the valuation of coefficients.

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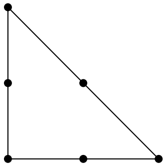

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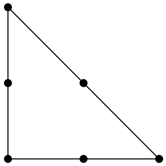

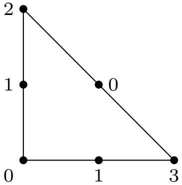

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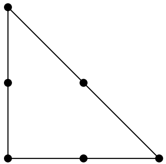

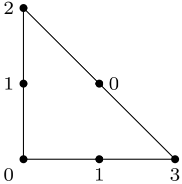

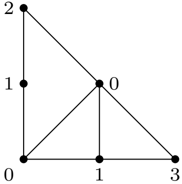

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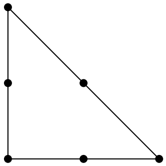
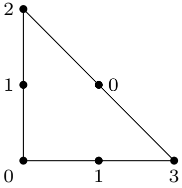

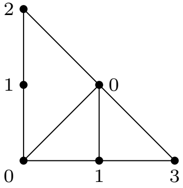

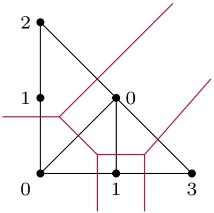

Uses of tropical geometry

- Enumerative geometry
- Computational algebraic geometry
- Matroid theory

Abstract tropical geometry

Tropical Curves - Abstract

Definition

A *stable abstract tropical curve* is a metric graph $\Gamma = (V, E, \ell, \gamma)$

- vertices V,
- edges E, which contains both finite edges and half-rays we call *ends*,
- a length function $\ell : E \to \mathbb{R}_{>0} \cup \{\infty\}$
- a genus function $\gamma : V \to \mathbb{Z}_{\geq 0}$.
- each vertex has at least $3 - 2g$ outgoing edges.

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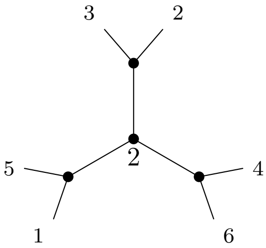
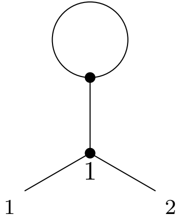
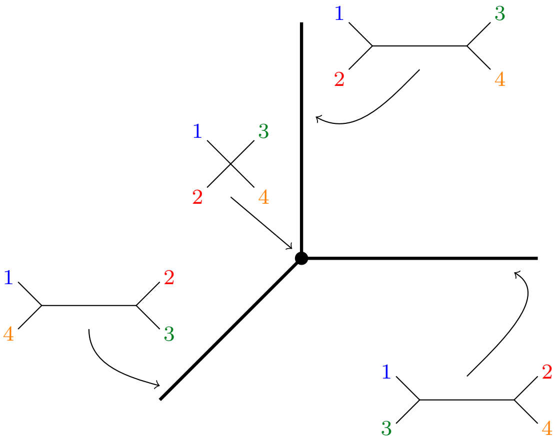


(a)



(b)

The moduli space of tropical curves



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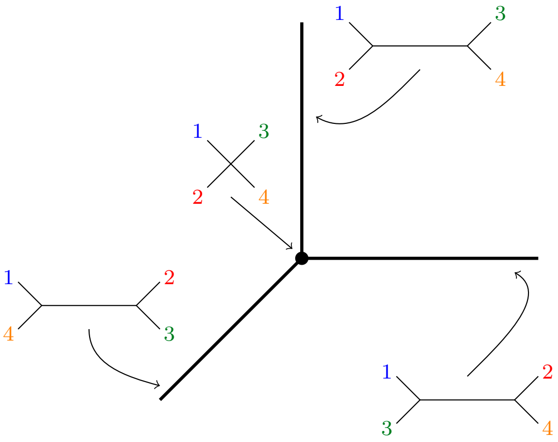

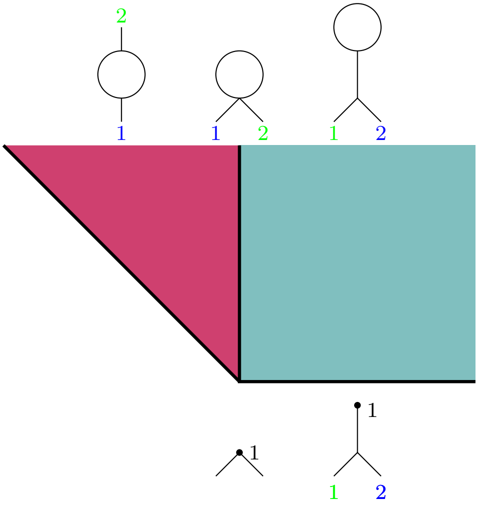

Tropical analogues of Teichmüller space

based on *Handlebodies, Outer Space, and Tropical Geometry*
[Ramadas-Silversmith-Vogtmann-Winarski]

$\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$

Definition

$\mathcal{M}_{g,n}$ is the parameter space of smooth projective irreducible algebraic curves (Riemann surfaces) of genus g with n labelled points. It is a (quasiprojective) variety of dimension $3g-3+n$.
Its *Deligne-Mumford compactification* is the space $\overline{\mathcal{M}}_{g,n}$

Theorem (Many)

$\mathcal{M}_{g,n}^{\text{trop}}$ is the tropical analogue of both $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. In particular, its structure gives a recipe to construct $\overline{\mathcal{M}}_{g,n}$

Boundary complexes

Definition

A boundary complex of a compact space X is a polyhedral complex whose polytopes parametrise the patterns of intersections in the boundary strata of X.

Example

$\mathrm{Link}(\mathcal{M}^{\mathrm{trop}_{g,n}})$ is the boundary complex of $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$.

Moduli of curves and Teichmüller spaces

Definition

The Teichmüller space $\mathcal{T}(S_{g,n})$ associated to an (oriented) Riemann surface $S_{g,n}$ is the space of Riemann surfaces of genus g with n points together with a marking given by a homeomorphism from $S_{g,n}$.
Its partial compactification is $\overline{\mathcal{T}}(S_{g,n})$, called the augmented Teichmüller space.

Theorem

$\mathcal{M}_{g,n}$ is the quotient of $\mathcal{T}(S_{g,n})$ by the mapping class group $\mathrm{MCG}(S_{g,n})$.
Analogously, $\overline{\mathcal{M}}_{g,n}$ is the quotient of $\overline{\mathcal{T}}(S_{g,n})$ by $\mathrm{MCG}(S_{g,n})$.

The curve complex

Definition

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Proposition (Ramadas '25)

The curve complex $\mathcal{C}(S_{g,n})$ is the boundary complex of $\overline{\mathcal{T}}(S_{g,n}) \setminus \mathcal{T}(S_{g,n})$.

Tropicalising the correlation

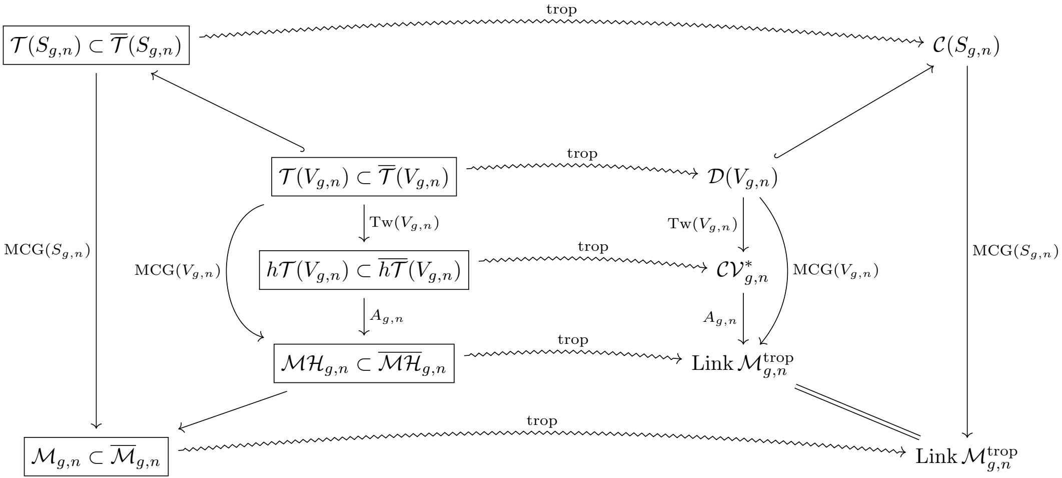
$$
\begin{array}{ccc}
\boxed{\mathcal{T}(S_{g,n}) \subset \overline{\mathcal{T}}(S_{g,n})} & \xrightarrow{\text{trop}} & \mathcal{C}(S_{g,n}) \\
\Big\downarrow \scriptstyle \mathrm{MCG}(S_{g,n}) & & \Big\downarrow \scriptstyle \mathrm{MCG}(S_{g,n}) \\
\boxed{\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}} & \xrightarrow{\text{trop}} & \mathrm{Link}\,\mathcal{M}^{\mathrm{trop}}_{g,n}
\end{array}
$$

Cheat sheet

$\mathcal{T}(S_{g,n})$	$\mathrm{MCG}(S_{g,n})$	$\mathcal{M}_{g,n}$
Teichmüller space	Mapping class group	Moduli space of curves

$\mathrm{Link}\,\mathcal{M}^{\mathrm{trop}_{g,n}}$	$\mathcal{C}(S_{g,n})$
Moduli space of tropical curves with edge-length 1	Curve complex of $S_{g,n}$

The bigger picture [Ramadas-Silversmith-Vogtmann-Winarski]



Our Observation

Graphs of measured foliations behave a lot like tropical Hurwitz covers.