COLLAR PARAMETERS FOR TEICHMÜLLER SPACE & MEASURED FOLIATIONS ON A SURFACE RESEARCH ANNOUNCEMENT

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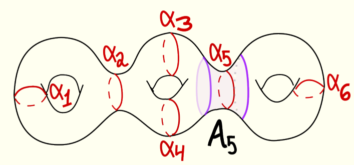
Abstract. We defined a new set of coordinates with respect to which the Thurston compactification of Teichmüller space is the radial compactification of Euclidean space.

The seminal work [7] of Thurston uses lengths of simple closed curves on a surface to define a compactification of its Teichmüller space. Let $\mathcal{S}$ be the set of isotopy classes of essential simple closed curves on a closed orientable surface Σ of genus $g \geq 2$. Throughout this paper a *measured foliation* is a *transversally measured singular foliation on a surface.* A hyperbolic metric and a measured foliation on Σ each assign a length to each element of $\mathcal{S}$. Both then determine a projectivized length function on $\mathcal{S}$, leading to Thurston's famous [7] compactification of the Teichmüller space $\mathcal{T}(\Sigma)$.



As depicted to the right, let $\{\alpha_i\}$ denote a set of pairwise disjoint simple closed curves on Σ whose complement is a disjoint union of 3-holed spheres (pants). Fenchel-Nielsen coordinates on $\mathcal{T}(\Sigma)$ assign a length and twist parameter to each α_i. Dehn-Thurston coordinates use similar data to parameterize measured foliations. In both cases the length is positive, or possibly zero in the case of measured foliations, but the twist is an arbitrary real number. The *collar parameter (CP) coordinates* we define here assign a point in $\mathbb{R}^2$ to each α_i. They are a variant of the Fenchel-Nielsen and Dehn-Thurston coordinates (the relationship is in Proposition 4.1), and encode both the length and the twist parameters.

Using the CP coordinates on a Teichmüller space, the Thurston compactification is just the radial compactification of Euclidean space. The same result does not hold using Fenchel-Nielsen coordinates, as is apparent by considering sequences where all twist parameters are zero and the lengths of the α_i go to zero at various rates. For such a sequence the parameters converge to the origin, and the limit in the Thurston boundary depends on the direction of approach to the origin.

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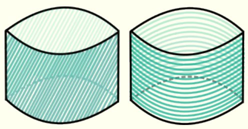
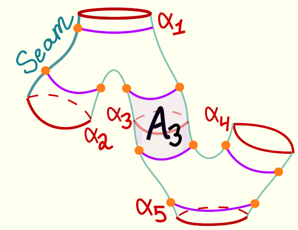
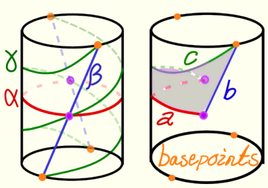
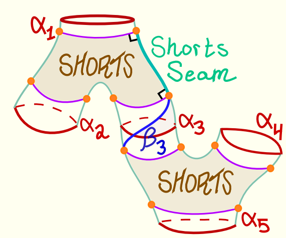


We now describe CP coordinates. In what follows the term *structure* means either a hyperbolic metric or a measured foliation on Σ. Each α_i is contained in an annulus A_i that in some sense is maximal in the structure. In a hyperbolic structure, A_i is provided by the *Collar Lemma* [1]. For measured foliations, Proposition 1.4 shows that each measured foliation is equivalent to one such that either ∂A_i is transverse to the foliation (upper left image), or else A_i is a union of smooth closed leaves such that the union of the closed leaves isotopic into A_i, that are in the complement of A_i, have zero transverse measure (upper right image). Such a maximal annulus in either structure is called a *standard collar*.



Given a structure on Σ and pair of pants in a pants decomposition, there is a structure-preserving reflection of the pants whose fixed point set consists of three arcs, one arc connecting each pair of boundary components. The arcs are called *seams*. Each annulus A_i has two basepoints on each of its boundary components given by the intersection with the seams of the pants decomposition (depicted using orange dots). CP coordinates parameterize structures on the annulus up to isotopy fixing the basepoints. We triangulate the annulus using these basepoints, two arcs β and γ connecting them, plus three meridian circles around the annulus (image to the left). The structure on the annulus is determined by the lengths of the sides of a triangle in this triangulation (a,b,c in the image). An equation relates these lengths, giving a parameter space $\mathbb{R}^2$ with coordinates that are certain linear combinations of edge lengths. This parameter space is the space of *collar parameters*, and is described in §1.2. It is a pleasant fact that for both structures the equation is symmetric in the three edge lengths.





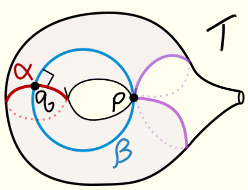
In the hyperbolic case, the curves $\{\alpha_i\}$ are geodesics and separate Σ into hyperbolic pants. Deleting the interiors of the annuli results in subsurfaces called *shorts*. The boundary components of the shorts are hypercycles (curves equidistant from a geodesic). Every point in the shorts is within distance $\cosh^{-1}(3)$ of the boundary.

The intersection of a seam of the pants with the shorts is the unique geodesic arc in the shorts connecting that pair of boundary components and meeting them orthogonally. These arcs are called the *seams* of the shorts. The shorts are determined up to isometry by the data for the annuli.

Gluing the shorts to the annuli so that the basepoints on the annuli are endpoints of seams (see §1.3) parameterizes Teichmüller space by a product of parameter spaces for the annuli.

A similar procedure works for measured foliations. Each measured foliation on the annulus is *linear* (see §1.5). The measured foliations on the shorts are determined solely by the measure (length) of the boundary components. Again there are seams: the fixed points of a reflection that preserves the transverse measure. These foliations yield a global parametrization of the space of measured foliations on Σ, as a product of the parameter spaces for the annuli.

This gives the parameterizations $\Theta_{\mathcal{T}} : \mathbb{R}^{6g-6} \to \mathcal{T}(\Sigma)$ of the Teichmüller space $\mathcal{T}(\Sigma)$, and $\Theta_{\mathcal{MF}} : \mathbb{R}^{6g-6} \to \mathcal{MF}(\Sigma)$ of the space $\mathcal{MF}(\Sigma)$ of measured foliations, using CP coordinates. Since both sets of coordinates are determined by the lengths of the *same sides of the same triangles* the homeomorphism $\Theta_{\mathcal{T}} \circ \Theta_{\mathcal{MF}}^{-1}$, sending a measured foliation to a hyperbolic metric, is a *good approximation* for large foliations. This works so well because most of the length of a geodesic (after a small perturbation), and all the measure of the measured foliation, is concentrated in the collars.



A hyperbolic torus with a geodesic boundary component can be obtained by identifying two boundary components of a hyperbolic pair of pants that have the same length (α in the image). The limit, as the length of the torus' boundary component goes to zero, is a complete hyperbolic once-punctured torus T with finite area 2π. Because of the choice of the depth of a standard collar, for any standard collar $\mathbb{A}$ containing a closed geodesic α, there is such a T that contains an isometric image of the interior of $\mathbb{A}$ and exactly two points, one on each (purple) boundary component of $\mathbb{A}$, are identified to a single point p in T. Twisting along α produces a one-parameter family of such tori. A geodesic arc crossing $\mathbb{A}$ and with these endpoints gives a geodesic loop β in T that contains p. There is another loop γ such that α, β, γ are three closed geodesics in T that pairwise-intersect precisely once. With suitable orientations, we have $\alpha \cdot \beta \cdot \gamma = 1 \in \pi_1 T$.

These closed geodesics contain the three edges of a triangle in the triangulation of $\mathbb{A}$, hence define the collar parameters. More precisely, the length of a triangle edge is half that of the geodesic loop containing it. The commutator $[\alpha, \beta]$ is parabolic. The formula for the trace of this parabolic, expressed in terms of the lengths of α, β, and γ, gives the equation relating the edge lengths in a standard collar. *One may thus regard the collar parameters of a standard collar as a point in the Teichmüller space of finite area complete hyperbolic metrics on T. The collar parameters are* *a projection of $\mathcal{T}(\Sigma)$ onto the product of $3g-3$ copies of $\mathcal{T}(T)$. Similarly for measured foliations. Thus the projection from $\mathcal{MF}(S)$ to $\mathcal{T}(S)$ factorises as a product of projections from $\mathcal{MF}(T)$ to $\mathcal{T}(T)$.*

A *quadratic differential* on a surface is a section of the symmetric square of the cotangent bundle. Thus a Riemannian metric is a quadratic differential that is a quadratic form of rank 2, e.g. $3dx^2 + dxdy + 5dy^2$. Also a measured foliation corresponds to a quadratic differential that is a quadratic form of rank ≤ 1, e.g. the quadratic differential $(2dx - 3dy)^2$ corresponds to a measured foliation with leaves $2x - 3y = const$. The term *holomorphic* is conspicuously absent.

Given $v \in \mathbb{R}^{6g-6}$ with $|v| \leq 1$ one can write down an explicit quadratic differential on Σ that varies continuously with v. When $|v| < 1$ this quadratic differential is a rescaling of a hyperbolic metric with collar parameters $v/(1-|v|)$. For $|v| = 1$ it is a measured foliation with collar parameter v. As such, the Thurston compactification of the Teichmüller space $\mathcal{T}(\Sigma)$ is realized as a subspace of the space of quadratic differentials $\mathcal{Q}(\Sigma)$ on Σ.

Theorem A bounds the difference between the length of an isotopy class of a loop in a hyperbolic metric and in the measured foliation corresponding to the metric via $\Theta_{\mathcal{MF}} \circ \Theta_{\mathcal{T}}^{-1}$. The bound is in terms of the minimum word length in the conjugacy class for the loop. This implies Theorem B concerning the compactification. These theorems follow from a stronger result that compares pointwise the hyperbolic metric and corresponding measured foliation after isotoping these structures into a nice position. This culminates in Theorem C, which lifts both a Teichmüller space and the corresponding space of measured foliations to spaces of quadratic differentials where the Thurston compactification arises from (rescaled) quadratic differentials, rather than isotopy classes of structures.

Some related work & further questions

In [4] Hensel and Sapir define a projection π from the space of filling geodesic currents on a closed surface Σ to its Teichmüller space.

Question 0.1. Can π be written in terms of the CP coordinates?

One can ask further questions as to whether the CP coordinates can simplify computations and give further insights into other structures related to a Teichmuller space, such as the tropical boundary [6] and modular structures [5] of Luo, and the cluster algebras of Fock-Goncharov [3].

1. Collar parameters

In what follows, Σ is a closed orientable surface of genus $g \geq 2$ and a *structure* is either a hyperbolic metric or a measured foliation on Σ.

To provide a common frame of reference for the structures, we fix a triangulation of a *standard annulus* as follows (Figure 1). Define the circle $S^1 := \mathbb{R}/2\mathbb{Z}$, and *standard annulus* $\mathbb{A} := S^1 \times [-1, 1]$.

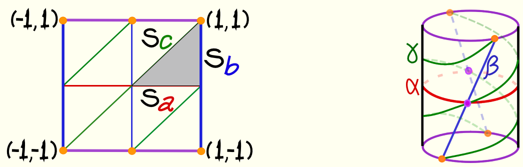


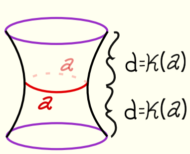
FIGURE 1. (a) The left image is a triangulation of a fundamental domain for the standard annulus $\mathbb{A} = S^1 \times [-1, 1]$ with the reference triangle shaded. The basepoints are in orange. (b) The right image shows α, β, γ. With a hyperbolic metric, the purple boundary curves are hypercycles.

The universal cover of the standard annulus is $\widetilde{\mathbb{A}} = \mathbb{R} \times [-1, 1] \subset \mathbb{R}^2$. Let $p : \mathbb{R} \times [-1, 1] \to \mathbb{A}$ be the covering map. The subset $[-1, 1]^2$ of $\widetilde{\mathbb{A}}$ is a fundamental domain. Triangulate $[-1, 1]^2$ with eight Euclidean triangles as shown in Figure 1a. Their images under p give a triangulation of $\mathbb{A}$. This triangulation contains a *reference triangle* with sides that are $s_a = p([0, 1] \times 0)$ and $s_b = p(1 \times [0, 1])$ and $s_c = p(\{(t, t) : 0 \leq t \leq 1\})$. There are two basepoints on each component of $\partial\mathbb{A}$; they are $p(1, \pm 1)$, and $p(0, \pm 1)$.

In the following we explain how a suitable triple (a, b, c) determines a hyperbolic metric or a measured foliation on the reference triangle. Using the triangulation of Figure 1, we then define, respectively, a hyperbolic metric or measured foliation on the standard annulus.

1.1. **Hyperbolic collars.**

A *hyperbolic collar* is an annulus A endowed with a hyperbolic metric so that A contains a simple closed geodesic α and each point of ∂A is a fixed distance d from α.

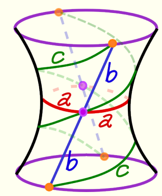


Thus ∂A consists of two hypercycles. The number d is called the *depth* of the collar. A hyperbolic collar whose core geodesic has length $2a$ is a *standard collar* if it has depth $\kappa(a) = \sinh^{-1}(1/\sinh(a))$.

The *Collar Lemma*, see [1], says that disjoint closed geodesics in a hyperbolic surface are contained in disjoint standard collars.

A hyperbolic collar will end up being standard if and only if the side lengths of the geodesic reference triangle in the triangulation just described satisfy the *collar equation*, as first introduced in Lemma 1.1:

Lemma 1.1 (collar equation)**.** Suppose S is a hyperbolic collar and the length of the core geodesic is $2a$. Then the edge lengths (a, b, c) of the geodesic reference triangle satisfy



$$\cosh^2 a + \cosh^2 b + \cosh^2 c = 2\cosh a \cosh b \cosh c. \tag{1}$$

if and only if S is standard.

In light of Lemma 1.1, we define the *collar equation*:

Definition 1.2 (collar equation)**.** We say a point in $\mathbb{R}^3$ satisfies the *collar equation* if it is in the set

$$\mathcal{H} = \{(a, b, c) \ : \ \cosh^2 a + \cosh^2 b + \cosh^2 c = 2\cosh a \cosh b \cosh c \quad \& \quad a, b, c > 0\}. \tag{2}$$

The set $\mathcal{H}$ is the left-hand image in Figure 2. It sits inside the cone from the origin on the triangle in the plane $x + y + z = 2$ with vertices the points $(1, 1, 0)$ and $(1, 0, 1)$ and $(0, 1, 1)$. It is asymptotic to the sides of this cone. The intersections of $\mathcal{H}$ with the planes $x + y + z = C$ are convex curves becoming larger and more nearly triangular as C increases. This is depicted on the right for $C = 3, 4, 5$.

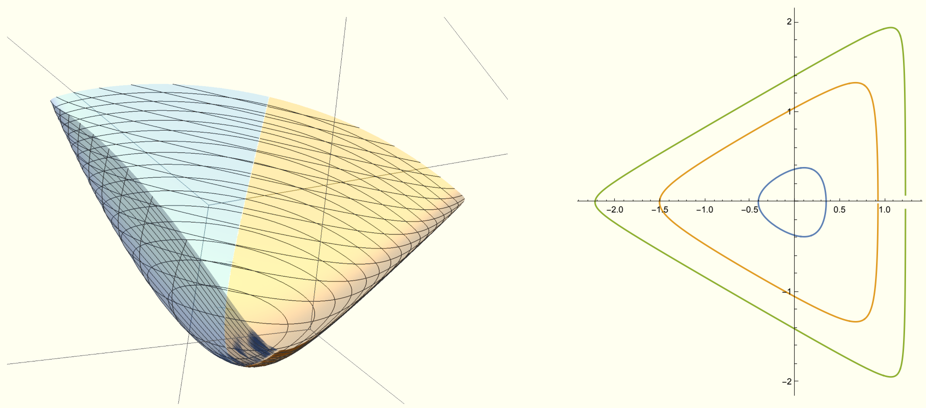
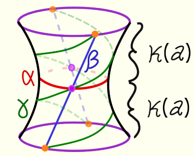
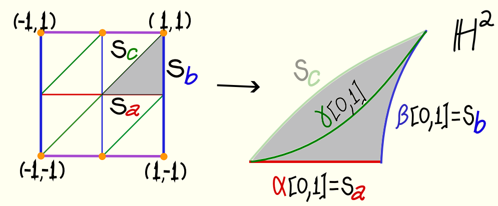


FIGURE 2. (a) on the left is $\mathcal{H}$ and (b) on the right are some cross sections of $\mathcal{H}$.



Given $(a, b, c) \in \mathcal{H}$ there is a unique hyperbolic metric $\mathbf{hyp}(a, b, c)$ on $\mathbb{A} = S^1 \times [-1, 1]$ satisfying simultaneously each of the following properties.

1. Endowed with the metric $\mathbf{hyp}(a, b, c)$, the annulus $\mathbb{A}$ is isometric to a standard collar with core curve of length $2a$.
2. The metric $\mathbf{hyp}(a, b, c)$ is preserved by rotation of the S^1 factor. In particular, the curve $\alpha : [-1, 1] \to \mathbb{A}$ given by $\alpha(t) = p(t, 0)$ is a constant speed geodesic.
3. The curve $\beta : [-1, 1] \to \mathbb{A}$ given by $\beta(t) = p(-1, t)$ is a geodesic of speed b and length $2b$.
4. There is a geodesic $\gamma : [-1, 1] \to \mathbb{A}$ of length $2c$ and homotopic rel endpoints to $t \mapsto p(t, t)$.



Rotational invariance implies each $S^1 \times y$ is a hypercycle and has constant speed. Half the length of each boundary component of $\mathbb{A}$ is $h = a \coth(a)$. The sides s_a and s_b of the reference triangle are hyperbolic geodesics, but the third side s_c is not. The *geodesic reference triangle* is the triangle in $\mathbb{A}$ with sides $\alpha[0, 1]$, and $\beta[0, 1]$, and $\gamma[0, 1]$. It has geodesic sides and is isotopic to the reference triangle without moving the vertices. It is clear that a standard triangulation of a standard collar can be isotoped without moving the basepoints to be the metric $\mathbf{hyp}(a, b, c)$ just described. We call such a metric *standard.*

The next result implies the distance between nearby points in a standard metric on an annulus differs from that given by a linear measured foliation by at most $\sqrt{2}$ times the Euclidean distance.

Lemma 1.3. The metric $\mathbf{hyp}(a, b, c)$ on $\mathbb{A}$ pulls back using the covering space projection p to the metric on $\widetilde{\mathbb{A}} = \mathbb{R} \times [-1, 1]$ given by

$$ds^2 = \left(a^2 + \left(\frac{a}{\sinh a}\right)^2 \left(\frac{\sinh(by)}{\sinh b}\right)^2\right) dx^2 \pm 2ab\sqrt{1 - \left(\frac{1}{\sinh a \sinh b}\right)^2} dxdy + b^2 dy^2.$$

The sign is $+1$ if $\cosh c \geq \cosh a \cosh b$. Hence

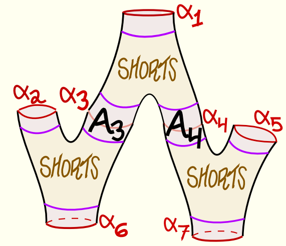
$$|ds^2 - (a.dx \pm b.dy)^2| \leq 2(dx^2 + dy^2).$$

1.2. **Triangle lengths and collar parameters.** Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by

$$\pi(a, b, c) = (4a - 2b - 2c,\ 2b - 2c).$$

This is the composition of orthogonal projection of $\mathbb{R}^3$ onto the subspace given by $a + b + c = 0$, followed by an isomorphism to $\mathbb{R}^2$. This particular isomorphism was chosen so that simple closed curves, thought of as measured foliations, map to integer points. The numbers (a, b, c) are called the *triangle lengths* and $(x, y) = \pi(a, b, c)$ are the *collar parameters*. It is routine to check that the map $\pi_{\mathcal{H}} = \pi|_{\mathcal{H}} : \mathcal{H} \to \mathbb{R}^2$ is a homeomorphism, so the collar parameters determine the triangle lengths.

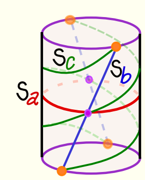
1.3. **Shorts decomposition.** Suppose $\mathcal{A} = \{\alpha_i : 1 \le i \le 3g - 3\}$ is a set of disjoint simple closed curves in Σ such that the closure of each component of $\Sigma \setminus \mathcal{A}$ is a pair of pants. Then there are $2g - 2$ complementary components.



Let $\mathbf{AN} = \{A_i : 1 \le i \le 3g - 3\}$ be a set of pairwise disjoint compact annuli in Σ such that A_i is a neighborhood of α_i for each i. Let $\mathbf{SH}$ be the closure of $\Sigma \setminus \bigcup \mathbf{AN}$. Then $\mathbf{SH} = \bigcup\{S_j : 1 \le j \le 2g - 2\}$, where each S_j is called a *pair of shorts*, and is a pair of pants with annular neighborhoods of the boundary components removed. $\mathcal{D} = (\mathbf{SH}, \mathbf{AN})$ is called a *shorts decomposition* of Σ.

Next we define a *marked shorts decomposition* by adding to the above a certain family of arcs

in the shorts S_j and annuli A_i. We choose some disjoint arcs in the shorts, called *seams*, such that there is exactly one seam connecting each distinct pair of boundary components in each shorts. The endpoints of the seams consist of two points on each boundary component of each A_i. These are the *basepoints*. Now, for each annulus A_i, choose a homeomorphism to the standard annulus $\mathbb{A}$ so that the basepoints are sent to vertices of the standard triangulation of the annulus. The *marking* on $\mathcal{D}$ consists of the seams in the S_j, together with the arcs s_a, s_b, s_c in each annulus A_i.



We emphasize that in this discussion there is no geometry: metric or measure. Subsequently we will put certain hyperbolic metrics onto Σ in such a way that the seams and s_a and s_b are all geodesic arcs, but the s_c are not geodesics.

1.4. **Parameterizing Teichmüller space.** The collar parameters determine hyperbolic metrics on the annuli and on the shorts. These fit together as dictated by the marked shorts decomposition. This determines a hyperbolic metric on Σ and a parameterization

$$\Theta_{\mathcal{T}}^{-1} : \prod \mathbb{R}^2 \longrightarrow \mathcal{T}(\Sigma).$$

1.5. **Parameterizing measured foliations.** Our point of view is that a measured foliation $|\omega|$ is determined by a 1-form ω. We consider two measured foliations *equivalent* that determine the same length functions on $\mathcal{S}$. If $\omega = 0$ on a subsurface then the foliation on that subsurface will not be important. The discussion below follows the terminology of [2, Section 6.2]. We wish to concentrate the transverse measure in the annuli. We will define a *standard measured foliation* on shorts and on an annulus. Fix a shorts decomposition of the surface Σ. Then a measured foliation on Σ is *standard* if the restriction to each shorts and each annulus is standard. It follows from [2] that:

Proposition 1.4. Every measured foliation on a surface is equivalent to a standard one.

Let P be shorts with boundary components $\delta_1, \delta_2, \delta_3$ (in [2] the corresponding boundary components are called $\gamma_1, \gamma_2, \gamma_3$). Given $m_1, m_2, m_2 \geq 0$ we define a *standard measured foliation* on P satisfying that the length of δ_i with respect to the transverse measure is m_i. Except for the case of $(m_1, m_2, m_3) = (0, 0, 0)$ the leaves of the foliation are shown in [2, Figure 6.6], but modified as follows. If $m_i = 0$ then an annular neigborhood of δ_i is foliated by smooth circles, and with transverse measure zero. The result is shown in Figure 3.

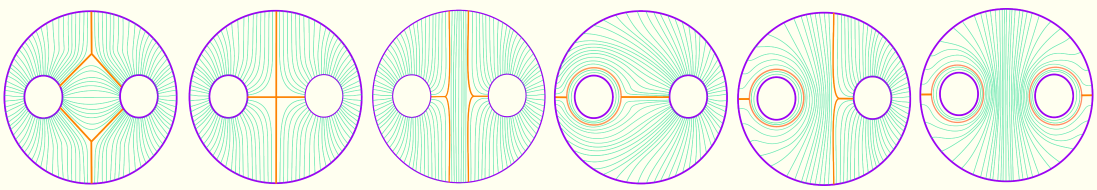


FIGURE 3. Possible foliations on the shorts [2, Figure 6.6]: from left to right these are where $m_1 + m_2 + m_3 > 2max\{m_1, m_2, m_3\}$, and $m_1 = m_2 + m_3$, and $m_1 > m_2 + m_3$, and $m_1 = m_3, m_2 = 0$, and $m_1 > m_3, m_2 = 0$, and $m_1 > 0, m_2 = m_3 = 0$.

The remaining case of $(m_1, m_2, m_3) = (0, 0, 0)$ is shown in Figure 4.

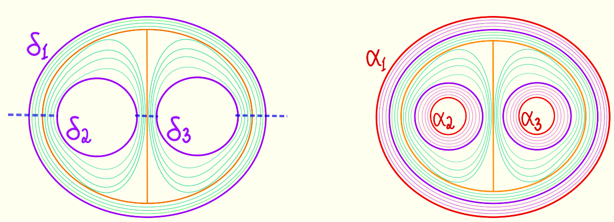


FIGURE 4. The measured foliation in the case of $m_1 = m_2 = m_3 = 0$ on shorts (left) and pants (right). The line of reflection in the shorts is the blue dotted line.

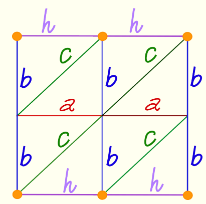
In each case there is an automorphism of the shorts P that is a reflection that fixes the union of three arcs, one connecting each pair of boundary components, and that is measure preserving.

It remains to define standard measured foliations on annuli. They are given by a linear 1-form. There are two kinds, depending on whether the leaves are circles, or arcs connecting the two boundary components. In the first case the transverse measure might be zero.

Definition 1.5 (triangle equality)**.** A point $v \in \mathbb{R}^3$ *satisfies the triangle equality* if it is in the set

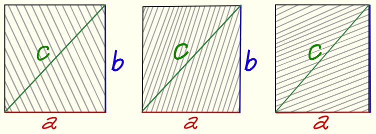
$$\Delta = \{(a, b, c) \;:\; a + b + c = 2\max\{a, b, c\} \quad \& \quad a, b, c \geq 0\}. \tag{3}$$

This is the cone from 0 on the boundary of a 2-simplex. Figure 2 shows how the subset $\mathcal{H}$ of $\mathbb{R}^3$ sits inside Δ like a hyperboloid inside its lightcone: they are asymptotic at infinity. Moreover $\pi|_\Delta : \Delta \to \mathbb{R}^2$ is a homeomorphism.



A measured foliation μ on $\mathbb{A}$ is *linear* if covered by a measured foliation $|df|$ on $\widetilde{\mathbb{A}} \subset \mathbb{R}^2$ given by the restriction of a, possibly identically zero, linear map $f : \mathbb{R}^2 \to \mathbb{R}$. A Euclidean line segment in $\mathbb{A}$ is either transverse to the foliation, or else contained in a leaf. We assign *lengths* (a, b, c) to the sides of the reference triangle in $\mathbb{A}$ by integrating $|df|$ along each side. Then $(a, b, c) \in \Delta$ and $h = a$. We again refer to $\pi(a, b, c) \in \mathbb{R}^2$ as *collar parameters*, and they determine these lengths for measured foliations.

Thus, as just described here and in §1.1, a collar parameter $p \in \mathbb{R}^2$ gives rise to both a measured foliation and a hyperbolic metric on $\mathbb{A}$.



Given edge lengths $v = (a, b, c) \in \Delta$, if $c \geq \max(a, b)$ then $c = a + b$ (leftmost foliation in the image), otherwise $c = |a - b|$ (middle is where $a = b + c$, right is where $b = a + c$). Let $\mathbf{mf}(v)$ be the linear measured foliation on $\mathbb{A}$ that assigns lengths a, b, c to the standard unit vectors $e_1, e_2, e_1 + e_2$ respectively. Then $\mathbf{mf}(v)$ lifts to a measured foliation $|\omega_v|$, where ω_v is the 1-form on $\mathbb{R}^2$ given by

$$\omega_v = \begin{cases} a.dx + b.dy & c = a + b \\ a.dx - b.dy & c = |a - b| \end{cases} \tag{4}$$

The figure to the left above shows a foliation on $\mathbb{R}^2$ such that ω_v vanishes on the tangent spaces of the leaves. When $v = 0$ we define the leaves to be the circles given by $\ker dy$.

An assignment of a point in $\pi(\Delta) = \mathbb{R}^2$ to each annulus in an annulus-shorts decomposition of Σ determines a measured foliation on Σ and yields a parametrization

$$\Theta_{\mathcal{MF}}^{-1} : \prod \mathbb{R}^2 \longrightarrow \mathcal{MF}(\Sigma).$$

1.6. **CP maps.** The *collar parameters* on a Teichmüller space and space of measured foliations are the maps $\Theta_{\mathcal{T}} = (\Theta_{\mathcal{T}}^{-1})^{-1} : \mathcal{T}(\Sigma) \to \mathbb{R}^{6g-6}$ and $\Theta_{\mathcal{MF}} = (\Theta_{\mathcal{MF}}^{-1})^{-1} : \mathcal{MF}(\Sigma) \to \mathbb{R}^{6g-6}$. The map $m = \Theta_{\mathcal{MF}}^{-1} \circ \Theta_{\mathcal{T}}^{-1} : \mathcal{T}(\Sigma) \to \mathcal{MF}(\Sigma)$ is called the *foliation map*, and the inverse $m^{-1} : \mathcal{MF}(\Sigma) \to \mathcal{T}(\Sigma)$ the *hyperbolization map*.

2. The Thurston compactification is the Radial Compactification

Suppose that ds is a positive semi-definite quadratic form on Σ. There is a *length function*

$$\mathcal{L}(ds) : \mathcal{S} \to \mathbb{R}$$

defined as follows. Given an element σ of $\mathcal{S}$ then

$$(\mathcal{L}(ds))(\sigma) = \inf \int_\gamma ds$$

where the infimum is taken over all simple closed curves γ in the isotopy class σ. We are interested in applying the length function to ds, when it is given either by a hyperbolic metric, or by a transversally measured foliation on Σ.

Choose a finite symmetric generating set $W \subset \pi_1(\Sigma)$. Set $W^1 = W$ and $W^{n+1} = \{x.y : x \in W,\ y \in W^n\}$. For $g \in \pi_1\Sigma$, define $w : \pi_1\Sigma \to \mathbb{Z}$ by $w(g) = \min\{n : \exists h \in \pi_1\Sigma \text{ with } hgh^{-1} \in W^n\}$. The number $w(g)$ is the *conjugacy* (or *cyclically reduced*) *word length* of g and is the minimum length on the alphabet W conjugate to g. If $v \in \mathbb{R}^{6g-6}$, then the hyperbolic structure, $\Theta_{\mathcal{T}}^{-1}(v)$, and measured foliation, $\Theta_{\mathcal{MF}}^{-1}(v)$, give length functions on $\mathcal{S}$ that differ by less than a fixed multiple of the word length:

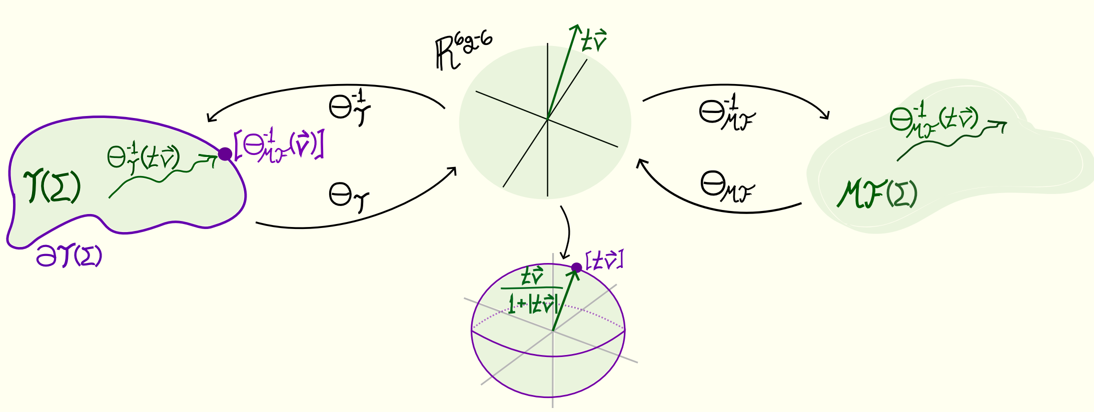
Main Theorem A. Given a conjugacy word length w on $\pi_1\Sigma$, there is a constant $C > 0$ such that the foliation map $m : \mathcal{T}(\Sigma) \to \mathcal{MF}(\Sigma)$ satisfies:

$$\forall\ \tau \in \mathcal{T}(\Sigma) \qquad |\ \mathcal{L}(\tau) - \mathcal{L}(m(\tau))\ | \leq C \cdot w.$$

Sketch proof. A hyperbolic geodesic can be isotoped to be a piecewise geodesic, δ, consisting of geodesic segments that are alternately in shorts and collars. Moreover δ is a (K, L)-quasi-geodesic with the constants K, L independent of the point in $\mathcal{T}(\Sigma)$. The length of each segment of δ in each shorts is uniformly bounded, since every point in a shorts is within $\cosh^{-1}(3)$ of the boundary. This relies on some beautiful properties of standard collars. The majority of the length of δ is in the collars. The number of segments in δ is bounded by a multiple of its conjugacy word length. Lemma 1.3 shows that on a standard collar the hyperbolic metric differs from a measured foliation by a bounded amount independent of the point in $\mathcal{T}(\Sigma)$. □

Using the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n$ given by $v \mapsto v/(1 + \|v\|)$, the *radial compactification* of $\mathbb{R}^n$ is the unit ball $B = \{v \in \mathbb{R}^n : \|v\| \leq 1\}$. The *Thurston compactification* is $\overline{\mathcal{T}}(\Sigma) = \mathcal{T}(\Sigma) \sqcup \mathbb{P}(\mathcal{MF})$. Since $\Theta_{\mathcal{MF}}^{-1}(tv) = t \cdot \Theta_{\mathcal{MF}}^{-1}(v)$ for each $t > 0$, it follows from Theorem A that the length functions $t^{-1}\mathcal{L}(\Theta_{\mathcal{T}}^{-1}(tv))$ converge to the length function of $\Theta_{\mathcal{MF}}^{-1}(v)$ provided $v \neq 0$.

Main Theorem B. Using $\Theta_{\mathcal{T}}$ coordinates to identify $\mathcal{T}(\Sigma) \equiv \mathbb{R}^{6g-6}$, the Thurston compactification is the radial compactification of $\mathbb{R}^{6g-6}$: if $0 \neq v \in \mathbb{R}^{6g-6}$, then $\lim_{t\to\infty} \Theta_{\mathcal{T}}^{-1}(tv) = [\Theta_{\mathcal{MF}}^{-1}(v)] \in \overline{\mathcal{T}}(\Sigma)$.



3. Realizing the compactification with quadratic differentials.

Suppose ds_0 is some Riemannian metric on Σ, not necessarily hyperbolic, called the *background metric*. We show that, *after a suitable isotopy*, a hyperbolic metric on Σ differs from some measured foliation on Σ by less than a fixed multiple of ds_0. Then integration along geodesics shows that the length functions with respect to the hyperbolic metric and measured foliation are close, provided these geodesics are not too long in the background metric. We formalize this with the following.

A seminorm ds on Σ is *C-efficient with respect to the background metric ds_0* if for each $g \in \pi_1\Sigma$ there is a ds-geodesic $\alpha : S^1 \to \Sigma$ that is freely homotopic to a loop representing g, and $\ell(\alpha, ds_0) \leq C \cdot w(g)$. A set of seminorms $\mathcal{N}$ is *uniformly efficient* if there is a $C > 0$ such that all the seminorms in $\mathcal{N}$ are C-efficient.

Since Σ is compact any two background metrics are bilipschitz equivalent. Thus whether or not a set of seminorms is uniformly efficient does not depend on the choice of background metric.

The space of quadratic differentials $\mathcal{Q}(\Sigma)$ on Σ contains the subspace, $\widetilde{\mathcal{T}}(\Sigma)$, of hyperbolic metrics and the subspace, $\widetilde{\mathcal{MF}}(\Sigma)$, of measured foliations. There are natural projections $\pi_{\mathcal{T}} : \widetilde{\mathcal{T}}(\Sigma) \to \mathcal{T}(\Sigma)$ and $\pi_{\mathcal{MF}} : \widetilde{\mathcal{MF}}(\Sigma) \to \mathcal{MF}(\Sigma)$, and $\pi_{\mathcal{T}}$ is a fiber bundle with fiber the group of diffeomorphisms isotopic to the identity. These maps have sections:

Main Theorem C (Efficient Realization Theorem)**.** Suppose Σ is a closed orientable surface with genus at least 2. Then there are embeddings $\widetilde{\Theta_{\mathcal{T}}^{-1}} : \mathbb{R}^{6g-6} \to \widetilde{\mathcal{T}}(\Sigma)$ and $\widetilde{\Theta_{\mathcal{MF}}^{-1}} : \mathbb{R}^{6g-6} \to \widetilde{\mathcal{MF}}(\Sigma)$ such that $\Theta_{\mathcal{MF}}^{-1} = \pi_{\mathcal{MF}} \circ \widetilde{\Theta_{\mathcal{MF}}^{-1}}$ and $\Theta_{\mathcal{T}}^{-1} = \pi_{\mathcal{T}} \circ \widetilde{\Theta_{\mathcal{T}}^{-1}}$, with $\widetilde{\Theta_{\mathcal{T}}^{-1}}(tv) = t\ \widetilde{\Theta_{\mathcal{T}}^{-1}}(v)$ for each $t \geq 0$ and $v \in \mathbb{R}^{6g-6}$. Moreover, given a background metric ds_0 on Σ, there is a $C = C(ds_0) > 0$ so that the image of $\widetilde{\Theta_{\mathcal{MF}}^{-1}}$ and of $\widetilde{\Theta_{\mathcal{T}}^{-1}}$ are uniformly efficient and

$$\forall v \in \mathbb{R}^{6g-6} \qquad |\ \widetilde{\Theta_{\mathcal{T}}^{-1}}(v) - \widetilde{\Theta_{\mathcal{MF}}^{-1}}(v)\ | \leq C \cdot |ds_0|. \tag{5}$$

To define $\widetilde{\Theta}_{\mathcal{MF}}^{-1}$ and $\widetilde{\Theta}_{\mathcal{T}}^{-1}$ involves writing down explicit metrics and measured foliations on shorts that match standard metrics on collars along the boundary.

4. Converting between Collar Parameters and Fenchel-Nielsen coordinates

We provide here the coordinate change maps between the CP coordinates we have defined, and the classical Fenchel-Nielsen coordinates on a Teichmüller space and Dehn-Thurston coordinates for the corresponding space of measured foliations.

Proposition 4.1. Suppose $(2\ell, 2\tau) \in \mathbb{R}^+ \times \mathbb{R}$ are the Fenchel-Nielsen coordinates of a point in Teichmüller space. [8]. Then the triangle lengths (a, b, c) are given by

$$\begin{aligned} a &= \ell, \quad \text{and} \\ b &= \cosh^{-1}(\cosh\tau\coth\ell), \quad \text{and} \\ c &= \cosh^{-1}(\cosh(\ell-\tau)\coth\ell). \end{aligned} \tag{6}$$

The collar parameters are given by

$$\begin{aligned} x &= 4\ell - 2\cosh^{-1}(\cosh\tau\coth\ell) - 2\cosh^{-1}(\cosh(\ell-\tau)\coth\ell) \quad \text{and} \\ y &= 2\cosh^{-1}(\cosh\tau\coth\ell) - 2\cosh^{-1}(\cosh(\ell-\tau)\coth\ell). \end{aligned} \tag{7}$$

If $(2\ell, 2\tau) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ are Dehn-Thurston coordinates of a point in the space of measured foliations [6] then the triangle lengths (a, b, c) are given by

$$\begin{aligned} a &= \ell, \quad \text{and} \\ b &= |\tau|, \quad \text{and} \\ c &= |\ell-\tau|, \end{aligned} \tag{8}$$

and the collar parameters by

$$\begin{aligned} x &= 4\ell - 2|\tau| - 2|\ell-\tau| \quad \text{and} \\ y &= 2|\tau| - 2|\ell-\tau|. \end{aligned} \tag{9}$$

If ℓ is large then $\coth\ell \approx 1$. Observe that replacing $\coth\ell$ by 1 in (6) yields (8).

5. The collar equation and Teichmüller space of a once-punctured torus

The collar equation is also the equation of the character variety for the Teichmüller space of finite area hyperbolic structures on a once-punctured torus T. This follows because the worst case for a standard collar is given by T, where there is a single self intersection point on the boundary of a standard collar. This point determines a reference triangle in T and this triangle determines the metric on T up to isotopy. Here are some of the details.

Refer to Figures 5 and 6. First we argue that for a right-angled finite area hyperbolic punctured torus that the depth of a maximal collar is given by κ. This is because the collar equation corresponds to the trace relation for the commutator as we explain below.

Let D be an ideal quadrilateral in $\mathbb{H}^2$ such that the common perpendiculars, A and B, to opposite sides of D are orthogonal. Then there is an $a \in \mathbb{R}_{>0}$ so that the lengths of these common perpendiculars are $2a$ and $2\kappa(a)$. Now D is a fundamental domain for a hyperbolic metric on T. This is obtained by identifying the opposite sides of D using isometries that translate along the common perpendiculars. The image of A in T is a simple closed geodesic α. The standard collar of α meets itself at one point p, that is the image of the endpoints of B. The image of B is a closed geodesic β on T orthogonal to α at a point q. There is a third closed geodesic γ on T containing p and r and homotopic to the product of α and β in $\pi_1(T, p)$. The point r is shown in Figure 5. Then the reference triangle has side lengths a, b, c that are the half-lengths of the geodesics α, β, and γ.

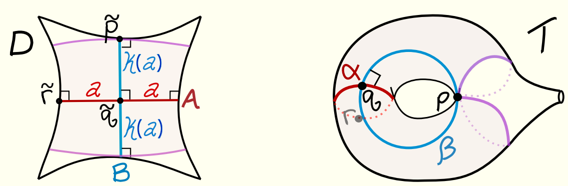


FIGURE 5. (a) The left image is an ideal quadrilateral D in $\mathbb{H}^2$ such that the common perpendiculars, A and B, to opposite sides of D are orthogonal. The two $\kappa(a)$-hypercycles are depicted in purple. (b) The right image is the punctured torus obtained by identifying the opposite sides of D using isometries that translate along the common perpendiculars. α is the image of A and β is the image of B. The image of the $\kappa(a)$-hypercycles correspond to the standard annulus boundary image.

If $A, B \in \mathrm{SL}(2, \mathbb{R})$ are the holonomies of α and β, respectively, then $C = AB$ is the holonomy of γ. The Markov-Fricke trace relation is

$$\operatorname{tr}[A, B] = (\operatorname{tr} A)^2 + (\operatorname{tr} B)^2 + (\operatorname{tr} C)^2 - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C - 2.$$

For a finite area punctured torus, and A, B as above, $\operatorname{tr}[A, B] = -2$. The relationship $\operatorname{tr} A = 2\cosh a$ between the trace and half the translation length now yields the collar equation in this case.

The hyperbolic structure on T has the property that α and β are orthogonal. Any finite area structure on T can be obtained from some such a pair α, β by an earthquake à la Fenchel-Nielsen along α by some distance. One can picture the triangulation on the resulting structure by cutting D along A and sliding the bottom half sideways, see Figure 6. The geodesic B is replaced by the geodesic connecting p and the image p' of p under the sideways slide. It follows that the standard collar of α in the resulting structure still has a single point of self intersection. Thus performing an earthquake does not change the depth of a maximal collar.

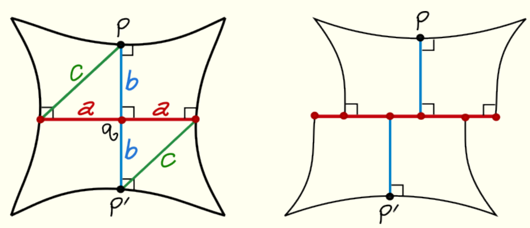


FIGURE 6. An earthquake does not change the maximal collar depth.

Proposition 5.1. Let T be a once-punctured torus and α, β a generating set for $\pi_1(T)$. Let $\mathcal{T}(T)$ denote the Teichmüller space of finite area hyperbolic metrics on T. Then there exists a homeomorphism $\mathfrak{H} : \mathcal{T}(T) \longrightarrow \mathcal{H}$ so that $\mathfrak{H}(\rho) = (a, b, c)$ are half the lengths of geodesic representatives of α, β, and $\alpha.\beta$ respectively.

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