Singularity of Cannon–Thurston maps

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Abstract

In a closed fibered hyperbolic 3-manifold M, the inclusion of a fiber S, with S and M lifted to the universal covers $\widetilde{S}$ and $\widetilde{M}$, gives an exponentially distorted embedding of the hyperbolic plane into hyperbolic 3-space. Nevertheless, Cannon and Thurston showed that there is a map from the circle at infinity of the hyperbolic plane to the 2-sphere at infinity of hyperbolic 3-space. The Cannon–Thurston map is surjective, finite-to-one, and gives a space-filling curve.

Here we use properties of geodesics to prove that many natural measures on the circle when pushed forward by the Cannon–Thurston map become singular with respect to many natural measures on the 2-sphere. The circle measures we consider are the Lebesgue measure and stationary measures that arise from fully supported random walks on the surface group. The measures on the sphere we consider are the Lebesgue measure and stationary measures that arise from geometric random walks on the 3-manifold group.

We obtain the singularity of measures from the following properties of typical geodesics. We prove that a hyperbolic geodesic sampled with respect to a pushforward measure asymptotically spends a definite proportion of its time close to a fiber. On the other hand, we show that a hyperbolic geodesic sampled with respect to a natural measure on the sphere spends an asymptotically negligible proportion of its time close to a fiber. For a more restricted class of circle measures, namely the Lebesgue measure and stationary measures from geometric random walks on the surface group, we also prove an effective result for the proportion of time spent close to a fiber.

Contents

1 Introduction

Let M be a closed 3-manifold that fibers over the circle. Suppose that the fiber S is a closed orientable surface. Fixing a fiber, M is homeomorphic to a mapping torus $S \times [0,1]/\sim$, where $\sim$ is the identification of $S \times \{0\}$ with $S \times \{1\}$ by a diffeomorphism $f\colon S \to S$. The topology of M depends only on the isotopy class of f, that is, only on f as a mapping class.

Thurston's hyperbolization theorem states that such a manifold M admits a complete hyperbolic metric if and only if S has negative Euler characteristic and the monodromy $f : S \to S$ is pseudo-Anosov. In this case, the universal covers of S and M can be identified with $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. The natural inclusion of a fiber as $S \times \{0\}$ induces a map $\iota : \widetilde{S} = \mathbb{H}^2 \hookrightarrow \mathbb{H}^3 = \widetilde{M}$ between the universal covers. The inclusion ι is exponentially distorted for the hyperbolic metrics on the source and the target. Cannon–Thurston proved that despite the distortion, the inclusion extends to a $\pi_1(S)$-equivariant, continuous map at infinity, known as the Cannon-Thurston map, which we shall also call ι, i.e. $\iota\colon \partial\mathbb{H}^2 = S^1_\infty \to S^2_\infty = \partial\mathbb{H}^3$ [CT07]. We maintain this notation of M, S, f, and ι throughout this paper. In particular, we assume throughout that $\chi(S) < 0$ and f is pseudo-Anosov.

We may compare measures on S^1_∞ with measures on S^2_∞, by taking the pushforward of the measures on S^1_∞ under the Cannon-Thurston map. We now describe the collections of measures that we will consider. The first collection, which we refer to as *surface measures*, are measures on the boundary of the universal cover of the surface S, namely S^1_∞. We write

S_h for S endowed with a particular choice of hyperbolic metric. That is, S_h is a hyperbolic surface.

Suppose that G is a group. A probability measure μ on G generates a random walk on G. The steps of the random walk are independent identically μ-distributed random variables $(g_n) \in (G, \mu)^{\mathbb{N}}$, and the location of the random walk at time n is given by $w_n = g_1 \dots g_n$. If G acts on a metric space (X, d) with a basepoint x_0, we say that μ has *finite exponential moment* if there is a constant $c > 1$ such that $\sum_{g \in G} \mu(g) c^{d(x_0, gx_0)}$ is finite. We say a probability measure μ on a group G is *geometric* if μ has finite exponential moment with respect to a word metric on G and if the support of μ generates G as a semigroup. We denote the set of geometric probability measures on G by $\mathcal{P}(G)$. In this paper, the group G will be either $\pi_1(S)$ or $\pi_1(M)$. In these cases, for any basepoint x_0, almost every sample path $w_n x_0$ of the random walk converges to the boundary, and the resulting boundary measure is known as the *hitting measure* determined by μ. For any two basepoints x_0 and x_1, the distance between $w_n x_0$ and $w_n x_1$ is constant, so the hitting measure does not depend on the choice of basepoint.

Definition 1. Let S_h be a closed hyperbolic surface of genus at least 2. Let S^1_∞ be the Gromov boundary of the universal cover, with basepoint x_0. We will refer to the following measures on S^1_∞ as *geometric surface measures.*

- The Lebesgue measure on S^1_∞ determined by the visual measure at x_0.
- The hitting measure on S^1_∞ for a random walk determined by a geometric probability distribution $\mu \in \mathcal{P}(\pi_1(S))$.

The Lebesgue measures that arise as the visual measures from different marked hyperbolic metrics are mutually singular, see for example Agard [Aga85]. However, for a fixed hyperbolic metric, different choices of basepoints give absolutely continuous measures. Similarly, different choices of the probability measure μ on $\pi_1(S)$ are expected to usually give mutually singular hitting measures, whereas for a fixed μ, the hitting measures arising from different basepoint choices are absolutely continuous with respect to each other.

Definition 1 covers a wide class of measures. In fact, it is a long standing conjecture of Guivarc'h–Kaimanovich–Ledrappier (see [DKN09, Conjecture 1.21]) that a hitting measure ν that arises from any finitely supported random walk on $\pi_1(S)$ is singular with respect to the Lebesgue measure on S^1_∞.

It will also be convenient to consider random walks on surface groups generated by more general probability measures. We say a probability measure μ on $\pi_1(S)$ is *nonelementary* if the semigroup generated by the support of μ contains a pair of non-trivial elements with disjoint fixed points on the boundary S^1_∞. We say a probability measure μ on $\pi_1(S)$ is *full* if every open set in S^1_∞ has positive hitting measure.

Definition 2. Let S_h be a closed hyperbolic surface of genus at least 2. Let S^1_∞ be the Gromov boundary of the universal cover. We refer to the larger collection of measures on S^1_∞ which contains in addition to geometric surface measures,

- the hitting measure on S^1_∞ for a random walk determined by a nonelementary, full probability measure μ on $\pi_1(S)$.

as the *full surface measures.*

The final collection of measures which we shall refer to as 3-*manifold measures*, are measures on the boundary of the universal cover of the 3-manifold M, namely S^2_∞.

Definition 3. Let M be a closed hyperbolic 3-manifold. Let S^2_∞ be the Gromov boundary of the universal cover, with basepoint x_0. We shall call the following measures 3-*manifold measures.*

- The Lebesgue measure on S^2_∞ determined by the visual measure at x_0.
- The hitting measure on S^2_∞ determined by a geometric random walk generated by $\mu \in \mathcal{P}(\pi_1(M))$.

As with the Lebesgue measures on S^1_∞, Lebesgue measures on S^2_∞ are expected to be mutually singular with respect to the hitting measures arising from finitely supported random walks on $\pi_1(M)$. The Lebesgue measures arising from different basepoints are absolutely continuous with respect to each other. The hitting measures arising from different probability measures μ are typically expected to be mutually singular.

Let M be a closed hyperbolic 3-manifold which fibers over the circle. As mentioned before, this determines a collection of 3-manifold measures on S^2_∞ given by Definition 3. Let S be a fiber and let S_h be a hyperbolic structure on S. These choices of M and S_h determine a collection of surface measures on S^1_∞, given by Definition 2. The pushforwards of these surface measures by the Cannon-Thurston map ι give measures on S^2_∞, so that we may now compare the surface measures to the 3-manifold measures:

Theorem 4. *Suppose that M is a closed hyperbolic 3-manifold that fibers over the circle. Then pushforwards to S^2_∞ under the Cannon-Thurston map of any full surface measure from Definition 2 (using any hyperbolic structure S_h on the fiber S) is mutually singular with respect to any 3-manifold measure from Definition 3.*

Suppose now that M is a compact hyperbolic 3-manifold and S is an incompressible surface in M. As an immediate corollary of Theorem 4, we obtain:

Corollary 5. *Suppose that μ is a geometric probability measure on the fundamental group of any incompressible surface S of a closed hyperbolic 3-manifold M. Then the hitting measure on S^2_∞ arising from μ is mutually singular with respect to both the Lebesgue measure on S^2_∞ and any hitting measure arising from a geometric random walk on $\pi_1(M)$.*

Proof. An incompressible surface S in M is either quasi-Fuchsian or a virtual fiber, by the Tameness Theorem [CG06] and the Covering Theorem [Can96]. If it is quasi-Fuchsian then the hitting measure is supported on the limit set of the surface subgroup. This limit set has measure zero for both the Lebesgue measure on S^2_∞ and for any hitting measure arising from a geometric random walk on $\pi_1(M)$.

On the other hand, if S is a virtual fiber then M admits a fibered finite cover in which the fiber F is a finite cover of S. As $\pi_1(F)$ is a finite index subgroup of $\pi_1(S)$, a geometric random walk on $\pi_1(S)$ is recurrent on $\pi_1(F)$, and the restriction of the random walk on $\pi_1(S)$ to $\pi_1(F)$ is the random walk on $\pi_1(F)$ generated by the first hitting measure μ' of the random walk of $\pi_1(S)$ on $\pi_1(F)$. The probability measure μ' is not finitely generated, but is nonelementary, and has the same hitting measure as μ, so is full. Therefore, Theorem 4 implies that the hitting measure is mutually singular with respect to either of the 3-manifold measures on S^2_∞. $\square$

Singularity of the pushforward of the Lebesgue measure on S^1_∞ with respect to the Lebesgue measure on S^2_∞ was previously shown by Tukia (see [Tuk89, page 430]) using conformal techniques, such as cross-ratios. This result also follows from work of Kim and Oh [KO24], showing singularity of conformal measures for representations of divergence type and Anosov representations, see their discussion of Cannon-Thurston maps in [KO24, Section 1]. Kim and Zimmer give further such rigidity results [KZ25], which also imply this result. The singularity of the push forward of the surface random walk hitting measure and the Lebesgue measure on S^2_∞ follows from Blachère, Haïssinsky and Mathieu, [BHM11, Proposition 5.5], which implies that the hitting measure has strictly smaller Hausdorff dimension, as the fundamental group of the surface is not quasi-isometrically embedded in $\mathbb{H}^3$. In light of the Guivarc'h–Kaimanovich–Ledrappier conjecture, we do not expect hitting measures to be conformal.

As we describe in the next section, Section 1.1, our approach uses the behavior of typical geodesics chosen according to the measures on the boundary which enables us to provide a unified (geometric) context in which the different types of surface measures can be compared to the 3-manifold measures.

1.1 Statistics for typical geodesics

An oriented geodesic in $\mathbb{H}^n$ is determined by its (ordered) endpoints in $(S^{n-1}_\infty \times S^{n-1}_\infty) \setminus \Delta$, where Δ is the diagonal. A probability measure ν on S^{n-1}_∞ gives rise to a probability measure $\nu \times \nu$ on the space of oriented geodesics in $\mathbb{H}^n$, as long as the diagonal has measure zero.

For random walks, if μ is not symmetric, we will use the measure $\nu \times \check{\nu}$ instead of the product measure, where $\check{\nu}$ is the hitting measure corresponding to the limits $\lim_{n\to-\infty} w_n x_0$ of the sample paths. To prove Theorem 4, we show the desired measures are mutually singular by showing that they give rise to typical geodesics possessing different behavior.

The fibration of M by closed surfaces lifts to a fibration of the universal cover by the universal covers of the fibers. We will call the image of $\widetilde{S}$ under the Cannon-Thurston map the *base fiber* $S_0 := \iota(\widetilde{S})$.

- With respect to the pushforwards of the surface measures to S^2_∞, almost all geodesics in $\mathbb{H}^3$ spend a positive proportion of time close to the base fiber S_0.
- With respect to the 3-manifold measures on S^2_∞, for almost all geodesics in $\mathbb{H}^3$, the proportion of time the geodesic spends close to the base fiber S_0 tends to zero.

We now give precise versions of these statements. Let $\gamma(t)$ be a geodesic with unit speed parametrization. We will write $\gamma([0,T])$ for the segment of γ between $\gamma(0)$ and $\gamma(T)$. For any subset A of γ, we will write $|A|$ for the standard Lebesgue measure of A. As M fibers over the circle, the universal cover also fibers as $\widetilde{S} \times \mathbb{R}$. As we explain in detail in Section 2.3, Cannon–Thurston define a pseudometric on $\widetilde{S}_h \times \mathbb{R}$ called the *Cannon–Thurston* metric. The Cannon–Thurston metric depends on a choice of hyperbolic metric S_h on S and on the pseudo-Anosov f in terms of its pair of invariant measured laminations Λ_+ and Λ_-. We will write (S_h, Λ) to denote a hyperbolic metric on S and the corresponding pair $\Lambda = (\Lambda_+, \Lambda_-)$ of invariant measured laminations on S_h. For full surface measures we show:

Theorem 6. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic structure on S together with a pair of invariant measured laminations. Let $\widetilde{S}_h \times \mathbb{R}$ be the universal cover of the corresponding mapping torus with the Cannon–Thurston metric, and let ι be the Cannon–Thurston map. Let ν be a full surface measure from Definition 2.*

Then there are constants $R \geqslant 0$ and $\epsilon > 0$ (that depend on f and ν) such that for $\iota_*\nu$-almost all geodesics γ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\gamma(t)$,

$$\lim_{T\to\infty} \frac{1}{T} |\gamma([0,T]) \cap N_R(S_0)| \geqslant \epsilon.$$

In fact, the constant ϵ tends to one as R tends to infinity. By restricting to geometric surface measures from Definition 1, we prove the following effective bound on the rate of convergence.

Theorem 7. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic structure on S together with a pair of invariant measured laminations. Let $\widetilde{S}_h \times \mathbb{R}$ be the universal cover of the corresponding mapping torus with the Cannon–Thurston metric, and let ι be the Cannon–Thurston map. Let ν be a geometric surface measure from Definition 1.*

Then there are constants $K > 0$ and $\alpha > 0$ such that for $\iota_*\nu$-almost all geodesics γ in $\widetilde{S}_h \times \mathbb{R}$ and for any unit speed parametrization $\gamma(t)$, for any $R \geqslant 0$,

$$\lim_{T\to\infty} \frac{1}{T} |\gamma([0,T]) \cap N_R(S_0)| \geqslant 1 - Ke^{-\alpha k^R}.$$

For 3-manifold measures we show:

Theorem 8. *Suppose that $f \colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic structure on S together with a pair of invariant measured laminations. Let $\widetilde{S}_h \times \mathbb{R}$ be the universal cover of the corresponding mapping torus with the Cannon–Thurston metric and let ι be the Cannon–Thurston map. Let ν be a 3-manifold measure from Definition 3.*

Then for ν-almost all geodesics γ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\gamma(t)$ and any $R > 0$,

$$\lim_{T \to \infty} \frac{1}{T} |\gamma([0,T]) \cap N_R(S_0)| = 0.$$

The mutual singularity of the surface measures and the 3-manifold measures, given in Theorem 4, is an immediate consequence of Theorem 6 and Theorem 8. Sets exhibiting the mutual singularity may be explicitly described as sets of geodesics which spend a positive proportion of time close to the base fiber S_0, and sets of geodesics whose proportion of time close to S_0 tends to zero.

Compared to Theorem 6, the effective version, namely Theorem 7, relies on an explicit construction of certain quasi-geodesics which we now briefly describe, see [GMPU25, Section 3] for more details. We discuss how this is related to previous constructions of quasigoedesics in Section 1.2.

For a compact hyperbolic 3-manifold M which fibers over the circle, the universal cover $\widetilde{M}$ has two natural structures. First, the hyperbolic metric on M lifts to a hyperbolic metric on $\widetilde{M}$, which is isometric to $\mathbb{H}^3$. Any $\pi_1(M)$-invariant metric on $\widetilde{M}$ will be quasi-isometric to this hyperbolic metric. Second, as M fibers over the circle, the fibration lifts to a product structure $\widetilde{S} \times \mathbb{R}$ on $\widetilde{M}$. This is only a topological product structure, as no product metric on $\widetilde{S} \times \mathbb{R}$ can be quasi-isometric to the hyperbolic metric.

Given a hyperbolic structure S_h, the pseudo-Anosov mapping class f determines a pair of invariant measured laminations. A choice of a hyperbolic structure S_h lifts to a hyperbolic structure $\widetilde{S}_h$ on the universal cover $\widetilde{S}$, and the invariant measured laminations can be realized as measured geodesic laminations in this metric. Cannon–Thurston [CT07] used the invariant measured geodesic laminations to construct a $\pi_1(M)$-invariant pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$, which is therefore quasi-isometric to the hyperbolic metric on $\widetilde{M}$, see Section 2.3 for further details. We call this pseudo-metric the *Cannon–Thurston metric* on $\widetilde{S}_h \times \mathbb{R}$.

Lifting an invariant lamination to $\widetilde{S}_h$, any leaf ℓ of the lift is geodesic with respect to the hyperbolic metric on $\widetilde{S}_h$. However, its image $\iota(\ell)$ in $\widetilde{S}_h \times \mathbb{R}$ is embedded metrically as a horocycle. In particular, $\iota(\ell)$ is not quasigeodesic, and a segment of $\iota(\gamma)$ of length L has endpoints distance roughly $\log L$ apart in the Cannon–Thurston metric on $\widetilde{S}_h \times \mathbb{R}$. For any surface measure, almost all geodesics γ in $\widetilde{S}_h$ have arbitrarily long subsegments that fellow travel leaves of the invariant laminations. The image $\iota(\gamma)$ in $\widetilde{S}_h \times \mathbb{R}$ is thus not quasigeodesic, even up to reparametrization. Our basic idea is to straighten "horocyclic" $\iota(\gamma)$ segments, that is, replace γ-subsegments that fellow-travel leaves of the invariant laminations by shortcuts. To do so, we define a *height function* $h_\gamma(t)$ along the geodesic, which is roughly $\log \log$ of the distance (in the unit tangent bundle) from the geodesic to the invariant laminations.

We then show that the paths $(\gamma(t), h_\gamma(t))$ in $\widetilde{S}_h \times \mathbb{R}$ are uniformly quasigeodesic, i.e. their quasigeodesic constants do not depend on the choice of the geodesic γ in $\widetilde{S}_h$.

In Section 2, we review some previous results we use, and define some notation. In Section 3, we prove Theorems 6 and 8, which immediately imply the singularity of measures result, Theorem 4. In Section 4, we prove the effective bounds in Theorem 7, by assuming the main result of [GMPU25, Section 3].

1.2 Quasigeodesics in Cannon–Thurston metrics

In this section, we discuss earlier constructions of quasigeodesics in Cannon–Thurston metrics, and some technical issues that led us to the variant that we use in this paper.

Given a subset A of the fiber, the union of all flow lines intersecting A is called the *ladder* over A. Cannon–Thurston [CT07] showed that ladders over leaves of the stable and unstable laminations are quasiconvex. Mitra [Mit98] extended this to show that ladders over geodesics in the fiber are quasiconvex, even in the more general context of hyperbolic group extensions. This implies that given a geodesic γ in the universal cover of the fiber surface, the geodesic in the universal cover of the 3-manifold connecting the endpoints of $\iota(\gamma)$ will be coarsely contained in the ladder over $\iota(\gamma)$, and has a parameterization in $\widetilde{S}_h \times \mathbb{R}$ as $(\iota(\gamma(t)), h_\gamma(t))$, for some function $h_\gamma \colon \mathbb{R} \to \mathbb{R}$. Note that *a priori* this function may depend on γ, and not necessarily arise from a function on the unit tangent bundle.

McMullen [McM01] constructed quasigeodesics in the classical Cannon–Thurston case, using saddle connections in the flat metric. Hamenstaedt [Ham05] considered the more general case of word hyperbolic extensions of surface groups, and Mj and Sardar [MS12] and Kapovich and Sardar [KS24] consider the general case of hyperbolic groups arising as hyperbolic-by-hyperbolic groups.

In all the above cases, the authors construct quasigeodesics by connecting segments of vertical flow lines by horizontal paths[1]. However, they do not prove an appropriate length upper bound for the vertical segments and the projection from the fiber geodesic to the quasi-geodesic by vertical flow to be coarsely distance non-increasing.

In proving the effective statements in Section 4, we need these properties, particularly the coarse distance non-increasing property. See Proposition 55. As we note in the example below, this property does not hold for all quasigeodesics, and depends on the details of the construction. For the example, consider a geodesic γ in the hyperbolic plane intersecting a horocycle h. See the illustration in in Figure 1 in the upper half space model of $\mathbb{H}^2$.

The two paths α and β in Figure 1 are quasigeodesics. For all horocycles h intersecting γ, the vertical projection from h onto β is uniformly coarsely distance non-increasing, that is, there are constants $D, K, C > 0$ such that for any horocycle h and any pair of points

[1] We warn the reader that other authors, for example Kapovich and Sardar [KS24], use the opposite convention for which directions are horizontal or vertical.

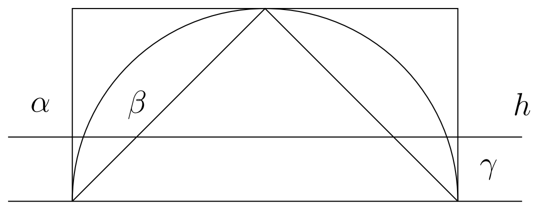

Figure 1: Two quasigeodesics in the upper half space model of $\mathbb{H}^2$.

distance at least D apart in the path metric on h, their projections to β are distance at most K times their distance along h plus C. However, the vertical projection from h to α does not have this property since a uniform D for h as its height (imaginary part) goes to zero.

In order to apply the ergodicity of the geodesic flow, we need to relate properties of quasigeodesics to a function on the unit tangent bundle of the surface. We expect it to be possible to modify the previous constructions of quasigeodesics to have the desired properties, though we can also show directly that the function we give on the unit tangent bundle gives quasigeodesic paths. In this paper we give detailed proofs of the singularity results and their effective versions, assuming the existence of quasigeodesics with the desired properties, stated precisely in Theorem 54 and Proposition 55. As verifying these properties is somewhat technical, we give the details in a separate paper [GMPU25].

1.3 Remarks on Related Work

It is interesting to compare Cannon–Thurston maps with the more classical space filling curves, such as the Peano curve. In contrast to our results here, the Peano curve is absolutely continuous.

The Peano curve is also Hölder with exponent 1/2. In contrast, Miyachi proved that Cannon–Thurston maps are not Hölder, see [Miy06, Theorem 1.1]. Since non-Hölder maps can be absolutely continuous, the absence of regularity does not imply singularity for the pushforward of the Lebesgue measure on the circle.

There are many Cannon–Thurston type phenomena generalizing the setup from fibered hyperbolic 3-manifolds. A natural generalization is given by Gromov hyperbolic groups that are extensions of surface groups, a fundamental group of a fibered hyperbolic 3-manifold being an extension by $\mathbb{Z}$. The existence of a Cannon–Thurston map is proved in [Mit98], and for more about the structure of the map see [MR18]. In this case, we expect pushforwards of stationary measures on the circle to be singular with respect to geometric/fully supported stationary measures on the Gromov boundary of the extension. On the one hand, we expect geodesics in the extension sampled by the pushforward measure to spend a definite proportion of their time in a neighborhood of the surface subgroup. On the other hand, we expect geodesics sampled by stationary measures resulting from geometric/fully supported random walks to spend asymptotically negligible time in a neighborhood of the surface subgroup.

While the reasons for our expectations are similar, our methods here are specific for hyperbolic 3-manifolds and these questions are left for now to future work. For a survey of other Cannon–Thurston type examples, see [GH24].

For Kleinian surface groups, one obtains Cannon-Thurston maps more generally from doubly degenerate surface group representations in $\mathrm{PSL}(2,\mathbb{C})$ [Mj14]. Without the $\mathbb{Z}$-periodicity, it makes good sense only to compare the pushforward of the Lebesgue measure on S^1_∞ with the Lebesgue measure on S^2_∞, and here the singularity of the pushforward is again covered by Tukia's work. We indicate which of our techniques underlying Theorem 6 and Theorem 8 hold when the $S \times \mathbb{R}$ has bounded geometry; we leave the full discussion of our perspective in the bounded geometry case to future work. We also leave more general Cannon–Thurston situations to future work.

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2 Background

In this section, we review some background and fix notation.

Let S be a closed orientable surface of genus at least two and suppose that $f\colon S \to S$ is an orientation preserving diffeomorphism. Then f determines a mapping torus $M_f = S \times [0,1]/\sim$, which is the quotient of $S \times [0,1]$ by the relation $(p,0) \sim (f(p),1)$. The map f is often referred to as the *monodromy map* for the mapping torus. We say a closed orientable 3-manifold M *fibers over the circle* if M is homeomorphic to a mapping torus M_f of some orientation preserving surface diffeomorphism $f\colon S \to S$. Thurston [Thu22] showed that the 3-manifold M_f is hyperbolic if and only if f is pseudo-Anosov. We will always assume that

f is pseudo-Anosov and so M_f is hyperbolic. For simplicity of notation, we will just write M for M_f.

The universal cover $\widetilde{M}$ may be thought of as $\mathbb{R}^2 \times \mathbb{R}$, where

- the universal cover $\widetilde{S}$ of the fiber S is identified with $\mathbb{R}^2$ and
- the monodromy map f acts by unit translation in the $\mathbb{R}$-factor direction.

Since f is pseudo-Anosov, the action of f on the Teichmüller space (of marked complete hyperbolic metrics on S) has an invariant axis on which f acts by translation. By identifying the $\mathbb{R}$ factor in $\widetilde{S} \times \mathbb{R}$ with the Teichmüller axis, we obtain a marked hyperbolic metric on S corresponding to $\widetilde{S} \times \{0\}$. We shall denote this metric by S_h. The monodromy map f acts by a change of marking of S_h. The universal cover $\widetilde{S_h}$ is isometric to the hyperbolic plane $\mathbb{H}^2$. We will write $d_{\mathbb{H}^2}$ for the hyperbolic metric on $\widetilde{S}_h$. The inclusion map ι sends $\widetilde{S}_h$ to $\widetilde{S}_h \times \{0\}$ in the universal cover $\widetilde{S}_h \times \mathbb{R}$ of M. We denote $\iota(\widetilde{S}_h)$ by S_0.

As in Section 1.1, the action of f on the fiber S_h has a pair of invariant measured geodesics laminations. The transverse measures are uniquely ergodic and the laminations are transverse to each other. The monodromy map f acts by stretching the leaves of one lamination (called the *unstable lamination*) and by contracting the leaves of the transverse lamination (called the *stable lamination*). The laminations can be lifted to $\pi_1(S)$-equivariant laminations of $\widetilde{S}_h$. These laminations reappear with further details in Section 2.1.

Using the lifted laminations, Cannon and Thurston [CT07] constructed a $\pi_1(M)$-invariant pseudometric on the universal cover $\widetilde{S}_h \times \mathbb{R}$ of M. See Section 2.3 for further details about the *Cannon–Thurston metric*.

The fiber lying on the invariant Teichmüller axis also carries a singular flat metric given by the associated quadratic differential on the underlying marked conformal surface. The monodromy map f acts affinely on the singular flat metric stretching the horizontal foliation and contracting (by the reciprocal of the stretch factor) the vertical foliation. We call the singular flat metric the *invariant flat metric* for f.

In an analogous way to Cannon–Thurston, we can use the invariant flat metric to construct a $\pi_1(M)$-equivariant metric on the universal cover $\widetilde{S}_q \times \mathbb{R}$ of M. This metric is known as the *singular solv metric*.

By the Švarc–Milnor lemma, both metrics, and also the hyperbolic metric, on $\mathbb{H}^3$ are quasi-isometric to $\pi_1(M)$ and so quasi-isometric to each other.

We do not use the singular solv metric directly, but we do discuss the relation between the Cannon-Thurston metric and the singular solv metric in Section 2.9. We shall always write $\widetilde{S}_h \times \mathbb{R}$ for the universal cover of M to emphasize that we are using the Cannon-Thurston metric and not the others.

2.1 Measured laminations

The properties of measured laminations that we present below are standard, see e.g. [CB88]. We present them in detail to keep our discussion self-contained.

A (possibly bi-infinite) geodesic on S_h is *simple* if it has no self-intersections. A *geodesic lamination* on S_h is a closed union of simple pairwise disjoint geodesics. A *transverse measure* on a geodesic lamination Λ is a positive measure dm defined on local transverse arcs to the leaves of Λ that

- is invariant under any isotopy preserving the transverse intersections with the leaves of Λ, and
- is positive and finite on any nontrivial compact transversals.

Such a measure lifts to a $\pi_1(S)$-invariant transverse measure on the pre-image of Λ in $\mathbb{H}^2$. By abuse of notation, we will denote the pre-image also by Λ and the lifted measure also by dm. A geodesic lamination equipped with a transverse measure is called a *measured lamination*. We will only consider measured laminations, and so we will often just write lamination to mean measured lamination.

We say a measured lamination is *filling* if there are no essential simple closed curves disjoint from the lamination. The complement of a filling lamination is a union of ideal polygons with finitely many sides. We say a leaf of the lamination is a *boundary leaf* if it is the boundary of an ideal polygon. There are finitely many ideal complementary regions in the compact surface S_h, and so there are only finitely many boundary leaves in S_h. This implies that there are countably many boundary leaves in the universal cover $\widetilde{S}_h$.

We say a measured lamination is *minimal* if every leaf is dense in the lamination. A minimal filling lamination has the following properties:

- there are uncountably many leaves,
- no leaf is isolated, and
- the transverse measure is non-atomic.

We say a pair of measured laminations Λ_+ and Λ_- *bind* S_h if each geodesic ray on S_h crosses a leaf of $\Lambda_+ \cup \Lambda_-$.

Definition 9. We shall write (S_h, Λ) for a triple consisting of a hyperbolic metric on a compact surface S, together with a pair of minimal filling measured laminations Λ_+ and Λ_- which bind the surface. We refer to such a triple (S_h, Λ) as a *hyperbolic surface and a full pair of laminations.*

A pseudo-Anosov map f determines a pair of invariant measured laminations called stable and unstable laminations, which we shall denote (Λ_+, dx) and (Λ_-, dy) respectively, where dx and dy are the transverse measures. In particular,

- $f(\Lambda_+) = \Lambda_+$ and $f_* dx = k dx$, and
- $f(\Lambda_-) = \Lambda_-$ and $f_* dy = k^{-1} dy$,

where $k = k_f > 1$ is known as the *stretch factor* of the pseudo-Anosov map. We shall often write Λ_+ or Λ_- to refer to the measured laminations if we do not need to refer to the respective measures. In the coordinate system described in Section 2.3, leaves of Λ_- correspond to lines parallel to the x-axis, and leaves of Λ_+ correspond to lines parallel to the y-axis.

Cannon and Thurston [CT07, Theorem 10.1] showed that the invariant measured laminations Λ_+ and Λ_- are each minimal and filling, and together they *bind* the surface S_h. In fact, pseudo-Anosov invariant laminations are uniquely ergodic, that is, the transverse measures are unique up to scale. In particular, the invariant measured laminations Λ_+ and Λ_- form a *full pair* of laminations for S_h.

We now record some useful properties of full pairs of laminations.

Proposition 10. *Let (S, Λ) be a hyperbolic surface and a full pair of laminations. Then Λ_+ and Λ_- have no leaf in common.*

Proof. Suppose that Λ_+ and Λ_- share a leaf ℓ. Since each lamination is minimal, the common leaf ℓ is dense in both laminations. This implies that $\Lambda_+ = \Lambda_-$. But then any ideal complementary region contains a geodesic ray disjoint from both laminations, contradicting the fact that Λ_+ and Λ_- bind the surface. □

In Proposition 11, we combine the compactness of S with the absence of common leaves to deduce the following properties: first, there is a lower bound on the angle at any point of intersection of a leaf of Λ_+ with a leaf of Λ_-, second, if a leaf of Λ_+ in $\widetilde{S}_h$ is disjoint from a lift of leaf of Λ_- in $\widetilde{S}_h$, then there is a lower bound on the distance between them. The second property implies that the ideal complementary regions of one lamination do not share (ideal) vertices with the ideal complementary regions of the other lamination. Finally, there is an upper bound on the length of a segment of a leaf which does not intersect the other lamination.

Suppose ℓ and ℓ' are leaves of Λ_+ and Λ_- (in $\widetilde{S}_h$) that intersect, creating two pairs of complementary angles. We define the *angle of intersection* to be the smallest of the two angles at the point of intersection.

Proposition 11. *Suppose that (S_h, Λ) is a hyperbolic surface together with a full pair of measured laminations. Then there exist constants $\alpha_\Lambda, \epsilon_\Lambda, L_\Lambda > 0$ such that each of the following properties holds for any pair of leaves (ℓ_+, ℓ_-) with ℓ_+ in Λ_+ and ℓ_- in Λ_-:*

(11.1) If ℓ_+ and ℓ_- intersect, then their angle of intersection is at least α_Λ.

(11.2) If ℓ_+ and ℓ_- are disjoint, the distance between any two lifts of ℓ_+ and ℓ_- in $\widetilde{S}_h$ is at least ϵ_Λ.

(11.3) Any segment of a leaf of one of the laminations of length at least L_Λ intersects a leaf of the other lamination.

(11.4) No ideal complementary region in $\widetilde{S}_h \setminus \Lambda_+$ has an ideal vertex in common with an ideal complementary region of $\widetilde{S}_h \setminus \Lambda_-$.

Proposition (11.4) follows directly from Proposition (11.2); we prove the remaining statements.

Proof of Proposition (11.1). Suppose that there is a sequence (ℓ_n^-, ℓ_n^+) of pairs of intersecting leaves whose angles of intersection tend to zero. By compactness of the unit tangent bundle $T^1(S_h)$, we may pass to a subsequence of pairs to assume that

- the points of intersection converge in S_h, and
- the angles of intersections at these points go to zero.

Since laminations are closed subsets, we deduce that the laminations contain a common leaf, a contradiction to Proposition 10. □

Proof of Proposition (11.2). Suppose there is a sequence of pairs (ℓ_n^+, ℓ_n^-) of disjoint leaves in Λ_+ and Λ_- such that the distance between them tends to zero. By using cocompactness of the $\pi_1(S)$ action on $\widetilde{S}_h$, we may assume that the closest points between the leaves lie in a compact region in $\widetilde{S}_h$. Hence, we may pass to a convergent subsequence of pairs and further assume that the tangent vectors (to the leaves) at these pairs also converge. Since laminations are closed subsets, the sequence of leaves limit to a common leaf of Λ_+ and Λ_-, a contradiction by Proposition 10. □

Proof of Proposition (11.3). Suppose there is a sequence of leaves ℓ_n in Λ_+, containing subsegments $\sigma_n \subset \ell_n$ of length $|\sigma_n| = L_n \to \infty$ of Λ_+, such that the segments σ_n do not intersect Λ_-. By cocompactness, and the fact that laminations are closed, the subsegments σ_n limit to a leaf ℓ of Λ_+ which does not intersect Λ_-. This implies that there is a geodesic ray asymptotic to ℓ, which is disjoint from both laminations, contradicting the fact that the laminations bind the surface S_h. The exact same argument works with Λ_+ and Λ_- interchanged. □

2.2 Flow sets and ladders

The mapping torus construction determines a flow on the universal cover $\widetilde{S}_h \times \mathbb{R}$, i.e. a continuous 1-parameter family of homeomorphisms given by $F_t(p, s) = (p, s + t)$, which we will call the *suspension flow*. We shall consider the $\mathbb{R}$ component of $\widetilde{S}_h \times \mathbb{R}$ as "vertical".

Suppose that A is a subset of $\widetilde{S}_h$. We define the *suspension flow set* $F(A)$ to be

$$F(A) = \bigcup_{z \in \mathbb{R}} F_z(A).$$

If $A = \{p\}$ is a point, then $F(p)$ is just the suspension flow line through p. If A is a hyperbolic geodesic γ in $\widetilde{S}_h$, then the suspension flow set $F(\gamma)$ is called the *ladder* of γ. We will refer to γ as the *base* of the ladder.

2.3 The Cannon-Thurston metric

Let (S_h, Λ) be a hyperbolic surface together with a full pair of laminations, and let $\widetilde{S}_h$ be the universal cover of S_h. Following [CT07], we define an infinitesimal pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$ by

$$ds^2 = k^{2z}dx^2 + k^{-2z}dy^2 + (\log k)^2 dz^2. \tag{1}$$

Here log will mean the natural log base e and we will write $\log_k$ for log base k. Throughout this paper, we will use $k > 1$ exclusively to refer to the constant in the definition of the Cannon-Thurston metric. If the pair of laminations is the pair of invariant laminations determined by a pseudo-Anosov map f, we will choose $k > 1$ to be the stretch factor of f. With this choice, the monodromy map acts on the universal cover by vertical translation by one unit. For an arbitrary full pair of laminations, we may choose $k = e$.

The infinitesimal pseudo-metric gives rise to a pseudometric $d_{\widetilde{S}_h \times \mathbb{R}}$ on $\widetilde{S}_h \times \mathbb{R}$ in the standard manner:

- integrating the pseudometric along rectifiable paths gives a pseudo-distance; and then
- defining the distance between two points as the infimum of the length over all rectifiable paths connecting the two points.

The resulting pseudo-metric on $\widetilde{S}_h \times \mathbb{R}$ is called the *Cannon–Thurston metric*. In the mapping torus case, the pseudo-metric is $\pi_1(M)$-invariant by construction. It is genuine pseudo-metric since if p and q are points in the same compact complementary region of $\widetilde{S}_h \setminus (\Lambda_+ \cup \Lambda_-)$, then for any $z_0 \in \mathbb{R}$, the corresponding points (p, z_0) and (q, z_0), with the same z-coordinate z_0, are distance zero apart in $\widetilde{S}_h \times \mathbb{R}$.

Theorem 12. [CT07, Theorem 5.1] *Let $f \colon S \to S$ be a pseudo-Anosov map, and let (S_h, Λ) be a hyperbolic metric on S together with a pair of invariant measured laminations for f, and let M be the corresponding fibered 3-manifold. Then the hyperbolic metric $d_{\mathbb{H}^3}$ and the $\pi_1(M)$–invariant global pseudometric $d_{\widetilde{S}_h \times \mathbb{R}} = \inf_\gamma \int_\gamma ds$ are quasi-isometric.*

If (S_h, Λ) is a full pair of laminations, then Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ is Gromov hyperbolic, as the vertical flow lines satisfy the flaring condition from the Bestvina-Feighn Combination Theorem [BF92].

Theorem 13. [BF92, page 88] *Let (S, Λ) be a hyperbolic metric on S together with a full pair of laminations. Then the Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ is Gromov hyperbolic.*

The Cannon-Thurston metric on $\widetilde{S}_h \times \mathbb{R}$ is quasi-isometric to $\mathbb{H}^3$ if and only if the pair of laminations have bounded geometry, by work of Rafi [Raf05]. In the bounded geometry case, the Cannon-Thurston metric is $\pi_1 S$-equivariantly quasi-isometric to $\mathbb{H}^3$, by the proof of the Ending Lamination Conjecture due to Minsky [Min94, Min10] and Brock, Canary and Minsky [BCM12].

The restriction of the infinitesimal pseudo-metric to the base fiber S_0 defines a pseudo-metric on $\widetilde{S}_h$. This metric is quasi-isometric to the hyperbolic metric $d_{\mathbb{H}^2}$, see Section 2.9 for further details.

Moving up in the z-direction expands distances in the x-direction and contracts them (by the reciprocal) in the y-direction. This is illustrated in Figure 2, where the horizontal lines are leaves of Λ_- and vertical lines are leaves of Λ_+.

The constant $\log k$ is chosen so that the map given by $(u, v) \mapsto (u, k^{-v})$ is an isometry from $\mathbb{R}^2$ with the metric $ds^2 = k^{2v}du^2 + (\log k)^2 dv^2$ to the upper half space model of $\mathbb{H}^2$ with the hyperbolic metric.

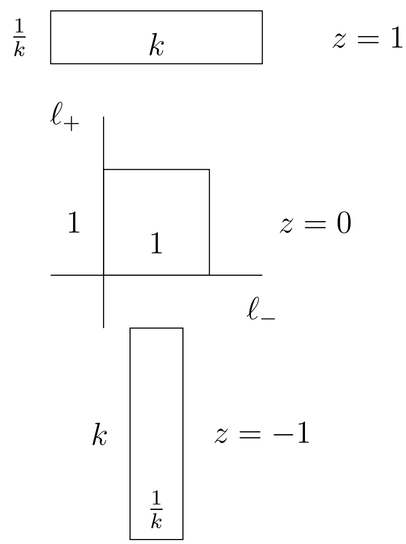

Figure 2: Rescaling arising from the vertical flow in the Cannon-Thurston metric.

Suppose that ℓ_+ is a leaf of Λ^+. The ladder $F(\ell_+)$ is then parametrized by the coordinates (y, z) in $\widetilde{S}_h \times \mathbb{R}$. By the definition of the pseudometric, $F(\ell_+)$ is a convex subset of $\widetilde{S}_h \times \mathbb{R}$. Moreover, $F(\ell_+)$ is quasi-isometric to $\mathbb{H}^2$, where in the upper half space model the quasi-isometry is given by $(y, z) \mapsto (y, k^{-z})$. The leaf ℓ_+ is a coarse horocycle, as is each image $F_z(\ell_+)$. The suspension flow lines are geodesics and the distance between two suspension flow lines decreases exponentially as the z-coordinate increases. Thus, as $z \to +\infty$, all suspension flow lines converge to the same limit point at infinity, as illustrated in Figure 3.

Similarly, if ℓ_- is a leaf of Λ_-, then the ladder $F(\ell_-)$ is parametrized by coordinates (x, z) in $\widetilde{S}_h \times \mathbb{R}$. The ladder $F(\ell_-)$ is again a convex subset of $\widetilde{S}_h \times \mathbb{R}$ quasi-isometric to $\mathbb{H}^2$, though in this case the quasi-isometry to the upper half space is given by $(x, z) \mapsto (x, k^z)$. The images of the leaf under the suspension flow, namely the $F_z(\ell_-)$, are coarse horocycles. The suspension flow lines are geodesics, and the distance between two suspension flow lines decreases exponentially as the z-coordinate decreases. Thus, as $z \to -\infty$, all suspension flow lines converge to the same limit point at infinity.

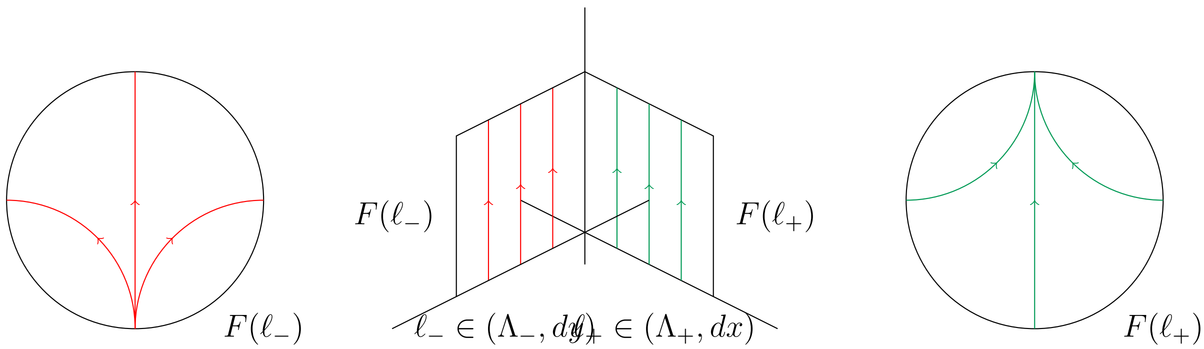

Figure 3: Ladders over leaves are quasi-isometric to $\mathbb{H}^2$.

In fact, ladders over arbitrary geodesics in $\widetilde{S}_h$ are quasiconvex, a special case of a more general result of Mitra [Mit98]. We will state Mitra's result using the notation of this paper.

Theorem 14. [Mit98, Lemma 4.1] *Suppose that f is a pseudo-Anosov map and $\widetilde{S}_h$ is a hyperbolic metric. Then, there is a constant K, such that for any geodesic γ in $\widetilde{S}_h$, the ladder $F(\gamma)$ is K-quasiconvex in $\widetilde{S}_h \times \mathbb{R}$.*

Ladders over arbitrary geodesics are also quasi-isometric to $\mathbb{H}^2$, though we do not use this fact directly.

2.4 Separation for ladders

Given two leaves ℓ and ℓ' of an invariant lamination, we define the distance between their ladders to be

$$d_{\widetilde{S}_h \times \mathbb{R}}(F(\ell), F(\ell')) = \inf \left\{ d_{\widetilde{S}_h \times \mathbb{R}}(p, p') \mid p \in F(\ell), p' \in F(\ell') \right\}.$$

In this section, we show that there is an $\epsilon = \epsilon_\Lambda > 0$ such that for any pair of ladders, the Cannon–Thurston distance between them is either zero, or at least ϵ. Furthermore, we prove that the limit sets in $S^2_\infty = \partial \mathbb{H}^3$ of any two ladders $F(\ell)$ and $F(\ell')$ intersect if and only if they are distance zero apart.

As the first step, we show that the distance is zero with the infimum realized if and only if the base of the ladders is a pair of leaves that are boundary leaves of a common ideal complementary region.

Proposition 15. *Suppose that (S_h, Λ) is a hyperbolic structure on S together with a full pair of measured laminations. Suppose that ℓ and ℓ' are leaves of the same lamination. Then there are points $p \in F(\ell)$ and $p' \in F(\ell')$ with $d_{\widetilde{S}_h \times \mathbb{R}}(p, p') = 0$ if and only if ℓ and ℓ' are boundary leaves of an ideal complementary region.*

Proof. Suppose that ℓ and ℓ' are boundary leaves of an ideal complementary region. Then we can find a subarc in $\widetilde{S}_h$ that connects ℓ and ℓ' such that its interior is disjoint from both laminations. The subarc then has length zero in the pseudo-metric, and so the distance between the ladders $F(\ell)$ and $F(\ell')$ is zero.

Conversely, suppose there is a path in $\widetilde{S}_h \times \mathbb{R}$ between $p \in F(\ell)$ and $p' \in F(\ell')$ that has zero length in the pseudo-metric. By the definition of the pseudo-metric, this path lies in a fiber $\widetilde{S}_h \times \{z\}$ and its interior is disjoint from the images (by the vertical flow F_z) of the laminations. This implies that ℓ and ℓ' are boundary leaves of a common ideal complementary region. □

For the remainder of this section, we set up some terminology. Suppose that ℓ and ℓ' are leaves of an invariant lamination. We denote the open strip in $\widetilde{S}_h$ with two boundary components ℓ and ℓ' by R. We denote by $A(R)$ the collection of arcs in $\widetilde{S}_h$ that have one endpoint on ℓ, the other endpoint on ℓ', and interior in R. The limit set of R is a disjoint union of two intervals, possibly with one of them a single point. We call these intervals I and I'. Let $\mathrm{Sep}_+(\ell, \ell')$ be the set of leaves in Λ_+ that separate ℓ from ℓ', that is, each leaf in $\mathrm{Sep}_+(\ell, \ell')$ is a leaf of Λ_+ that has one limit point in I, the other limit point in I'. Similarly, let $\mathrm{Sep}_-(\ell, \ell')$ of leaves in Λ_- that separate ℓ from ℓ'.

Lemma 16. *Suppose that ℓ and ℓ' are distinct leaves in Λ_+. Then $Sep_+(\ell, \ell')$ is non-empty if and only if ℓ and ℓ' are not boundary leaves of a common ideal complementary region of Λ_+.*

Proof. If $\mathrm{Sep}_+(\ell, \ell')$ is non-empty then ℓ and ℓ' cannot be boundary leaves of a common ideal complementary region of Λ_+.

Conversely, suppose that ℓ and ℓ' are not boundary leaves of a common ideal complementary region. Recall that I and I' are the two limit sets with one endpoint in ℓ, and the other endpoint in ℓ'. As the two geodesics ℓ and ℓ' are distinct, at least one of I and I' has non-empty interior.

If the interval I consists of a single point, its convex hull C_I is equal to I. If the interval I has non-empty interior, then the convex hull $C_I \subset R$ consists of the limit set I, together with all bi-infinite geodesics with both endpoints in I. In particular, the boundary of C_I is the bi-infinite geodesic α connecting the endpoints of I. Similarly, we denote by $C_{I'}$ the convex hull of I'. Again, if I' has non-empty interior, we denote by α' the bi-infinite geodesic connecting the endpoints of I'.

By convexity, any ideal complementary region of Λ_+ with all its ideal vertices in I is contained in C_I. Similarly, any ideal complementary region of Λ_+ with all its ideal vertices in I' is contained in $C_{I'}$.

The geodesics ℓ and ℓ', together with the geodesics α and α', form an ideal quadrilateral $T \subseteq R$ with at least three limit points, and so T has non-empty interior. Therefore, T must intersect an ideal complementary region U of Λ_+ contained in R. By construction, this complementary region has ideal vertices in both I and I'. We then find exactly two boundary leaves ℓ_1 and ℓ_2 of U connecting I to I'. If, as unordered pairs $(\ell_1, \ell_2) = (\ell, \ell')$, then ℓ and ℓ' are boundary leaves of a single ideal complementary region of Λ_+, a contradiction. We deduce that at least one of ℓ_1 or ℓ_2 is contained in $\mathrm{Sep}_+(\ell, \ell')$, and thus $\mathrm{Sep}_+(\ell, \ell')$ is non-empty, as required. □

By switching the invariant laminations, Lemma 16 also holds for Λ_- with $\mathrm{Sep}_-(\ell, \ell')$ non-empty if and only if ℓ and ℓ' are not boundary leaves of a common ideal complementary region of Λ_-.

Lemma 17. *Suppose that ℓ and ℓ' are leaves of an invariant lamination. Then the subset $Sep_+(\ell, \ell')$ is non-empty if and only if the dx-measure of $Sep_+(\ell, \ell')$ is positive. Similarly, $Sep_-(\ell, \ell')$ is non-empty if and only if the dy-measure of $Sep_-(\ell, \ell')$ is positive.*

Proof. If the dx-measure of $\mathrm{Sep}_+(\ell, \ell') > 0$ then $\mathrm{Sep}_+(\ell, \ell')$ is non-empty by the definition of transverse measure. Conversely, suppose that $\mathrm{Sep}_+(\ell, \ell')$ is non-empty and so there is a transverse arc contained in $A(R)$ that intersects in an interior point a leaf in $\mathrm{Sep}_+(\ell, \ell')$. Since an invariant lamination has no isolated leaves, it follows that such an arc has positive dx-measure. Since the dx-measure of the arc is a lower bound on the dx-measure of $\mathrm{Sep}_+(\ell, \ell')$, the lemma follows. □

Lemma 18. *Suppose that ℓ and ℓ' are leaves of an invariant lamination. Then $d_{\widetilde{S}_h \times \mathbb{R}}(F(\ell), F(\ell')) > 0$ if and only if the dx-measure of $Sep_+(\ell, \ell')$ and the dy-measure of $Sep_-(\ell, \ell')$ is positive.*

Proof. We let a be the dx-measure of $\mathrm{Sep}_+(\ell, \ell')$ and b the dy-measure of $\mathrm{Sep}_-(\ell, \ell')$. Suppose that γ is an arc in $A(R)$. Then γ must intersect in its interior every leaf in $\mathrm{Sep}_+(\ell, \ell')$ and every leaf in $\mathrm{Sep}_-(\ell, \ell')$. It follows that $dx(\gamma) \geqslant a$ and $dy(\gamma) \geqslant b$.

Suppose p and p' are points on ℓ and ℓ' respectively. By definition of the Cannon-Thurston metric, one of the two distances $d_{\widetilde{S}_h \times \mathbb{R}}((p, z), (p', z))$ or $d_{\widetilde{S}_h \times \mathbb{R}}((p, z'), (p', z'))$ is at most the distance $d_{\widetilde{S}_h \times \mathbb{R}}((p, z), (p', z'))$. Thus, we may assume that $z = z'$, that is, the pair of points are at the same height. Suppose that γ is an arc in $A(R)$ between p and p'. Then the length of $F_z(\gamma)$ equals $k^z dx(\gamma) + k^{-z} dy(\gamma) \geqslant k^z a + k^{-z} b$. The lemma now follows. □

We now show that the distance is zero but the infimum is not attained if and only if the ladders are over leaves that intersect a complementary region of the other lamination.

Proposition 19. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Suppose that ℓ and ℓ' are leaves of the same lamination and suppose that $d_{\widetilde{S}_h \times \mathbb{R}}(p, p') > 0$ for any pair of points $p \in F(\ell)$ and $p' \in F(\ell')$. Then the distance $d_{\widetilde{S}_h \times \mathbb{R}}(F(\ell), F(\ell')) = 0$ if and only if ℓ and ℓ' are not boundary leaves of a common ideal complementary region but intersect a common ideal complementary region of the other lamination.*

Proof. Breaking symmetry, we may assume that ℓ and ℓ' are leaves of Λ_+. The same argument holds for Λ_- by switching the laminations.

Suppose that $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell),F(\ell'))=0$ but $d_{\widetilde{S}_h\times\mathbb{R}}(p,p')>0$ for any pair of points $p\in F(\ell)$ and $p'\in F(\ell')$. By Proposition 15, ℓ and ℓ' are not boundary leaves of a common ideal complementary region of Λ_+. By Lemma 16, $\mathrm{Sep}_+(\ell,\ell')$ is non-empty. If $\mathrm{Sep}_-(\ell,\ell')$ is also non-empty, then by Lemma 18, $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell),F(\ell'))>0$, a contradiction. Thus $\mathrm{Sep}_-(\ell,\ell')$ is empty. This means that there is an arc γ in $A(R)$ with endpoints p on ℓ and p' on ℓ' such that the interior of γ intersects only Λ_+. But then γ is contained in a single ideal complementary region of Λ_-, as required.

Conversely, suppose that ℓ and ℓ' are not boundary leaves of a single ideal complementary region of Λ_+ but intersect a common ideal complementary region of Λ_-. By Lemma 16, $\mathrm{Sep}_+(\ell,\ell')$ is non-empty. Since any arc γ in $A(R)$ intersects $\mathrm{Sep}_+(\ell,\ell')$, the length of $F_z(\gamma)$ is at least k^z times the dx-measure of $\mathrm{Sep}_+(\ell,\ell')$. In particular, this implies that $d_{\widetilde{S}_h\times\mathbb{R}}(p,p')>0$ for any pair of points $p\in F(\ell)$ and $p'\in F(\ell')$. On the other hand, since ℓ and ℓ' intersect a common ideal complementary region of Λ_-, there is an arc γ in $A(R)$ with endpoints p on ℓ and p' on ℓ' such that the interior of γ intersects only Λ_+. Then the length of $F_z(\gamma)$ equals $k^z dx(\gamma)$ which goes to zero as $z\to-\infty$. Thus, $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell),F(\ell'))=0$, as required. □

Finally, we show the distance gap for ladders.

Proposition 20. *Suppose that (S_h,Λ) is a hyperbolic structure on S together with a full pair of measured laminations. Then there is a constant $\epsilon_3>0$ such that for any two leaves ℓ_1 and ℓ_2 of a lamination, either $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_1),F(\ell_2))\geqslant\epsilon_3$, or else $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_1),F(\ell_2))=0$.*

Proof. Suppose that there is a sequence of pairs of leaves ℓ_n and ℓ'_n of a lamination such that $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_n),F(\ell'_n))>0$ and tends to zero as $n\to\infty$. Breaking symmetry, we assume that ℓ_n and ℓ'_n are leaves of Λ_+. Since Λ_+ is a closed set, we may, by passing to a subsequence, assume that ℓ_n and ℓ'_n converge to leaves ℓ and ℓ' respectively. It follows that $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell),F(\ell'))=0$.

By Proposition 15 and Proposition 19, we can find an arc α in $A(R)$ with endpoints p on ℓ and p' on ℓ' such that either the interior of α is disjoint from both laminations, or the interior of α intersects only Λ_+.

Suppose that q_n on ℓ_n and q'_n on ℓ'_n are sequences of points that converge to p and p'. It follows that by choosing q_n and q'_n sufficiently close to p and p' we can find an arc α_n with endpoints q_n and q'_n such that the interior of α_n intersects only Λ_+. But then by Proposition 19, $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell_n),F(\ell'_n))=0$, a contradiction.

□

Finally, we show that the limit sets of two ladders intersect if and only if they are distance zero apart.

Proposition 21. *Suppose that (S_h,Λ) is a hyperbolic metric on S together with a full pair of measured laminations. For any leaves ℓ and ℓ' in a lamination, $\overline{F(\ell)}\cap\overline{F(\ell')}\neq\varnothing$ if and only if $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell),F(\ell'))=0$.*

Proof. Suppose that ℓ and ℓ' are in Λ_+ and $d_{\widetilde{S}_h\times\mathbb{R}}(F(\ell), F(\ell')) = 0$. By Proposition 15 and Proposition 19, ℓ and ℓ' are either boundary leaves of an ideal complementary region of Λ_+, or else intersect an ideal complementary region of Λ_-. It follows that there are points $p \in \ell$ and $p' \in \ell'$ and an arc α in $A(R)$ with endpoints p and p' such that the interior of α is either disjoint from both laminations or intersects only Λ_+.

If the interior is disjoint from both laminations then $F_z(p)$ and $F_z(p')$ are pseudo-metric distance zero for all z. Thus, the flow lines determine the same limit points at infinity.

Now suppose that the interior of α intersects Λ_+. Then $dx(\alpha) > 0$ and $dy(\alpha) = 0$. Then the distance between $F_z(p)$ and $F_z(p')$ is $k^z dx(\alpha)$ and hence the flow lines determine the same point at infinity as $z \to -\infty$.

Conversely, suppose $\overline{F(\ell)} \cap \overline{F(\ell')}$ is non-empty and let z_∞ be a point of the intersection. Every limit point in $\overline{F(\ell)}$ (similarly $\overline{F(\ell')}$) is a limit point of a suspension flow line, and hence there are points $p \in \ell$ and $p' \in \ell'$ such that their suspension flow lines $F(p)$ and $F(p')$ converge to z_∞ in one direction. The Cannon-Thurston metric is δ-hyperbolic and suspension flow lines are geodesics. We deduce that $F(p)$ and $F(p')$ are bounded distance in the direction of the common limit point. It follows that there is an arc β in $A(R)$ with endpoints p and p' such that the interior of β intersects only one of the laminations. Thus, the distance between $F(\ell)$ and $F(\ell')$ is zero, as desired. □

2.5 Quasigeodesics

We recall some basic facts about quasigeodesics, see for example Bridson and Haefliger [BH99, III.H].

Definition 22. Let (X, d) be a geodesic metric space and let $\gamma\colon I \to X$ be a path, where I is a (possibly infinite) connected subset of $\mathbb{R}$. Let $Q \geqslant 1$ and $c \geqslant 0$ be constants. The path γ is a *(Q, c)-quasigeodesic* if for all t_1 and t_2 in I,

$$\frac{1}{Q}|t_2 - t_1| - c \leqslant d(\gamma(t_1), \gamma(t_2)) \leqslant Q|t_2 - t_1| + c.$$

By [BH99, III.H Lemma 1.11], given a (Q, c)-quasigeodesic, there is a continuous (Q, c')-quasigeodesic with the same endpoints, so for our purposes we may assume that all quasi-geodesics are continuous, and we will do so from now on.

A *reparametrization* of a path $\gamma\colon \mathbb{R} \to X$ is the path $\gamma \circ \rho$, where $\rho\colon \mathbb{R} \to \mathbb{R}$ is a proper non-decreasing function. We say a path $\gamma\colon \mathbb{R} \to X$ is an *unparametrized* (Q, c)-quasigeodesic if there is a reparametrization of γ which is a (Q, c)-quasigeodesic.

We will use the following stability property for quasigeodesics in hyperbolic spaces, known as the Morse Lemma.

Lemma 23. [BH99, Theorem 1.7] *Let X be a δ-hyperbolic space. Then for any Q and c there is a constant L such that any (Q, c)-quasigeodesic is contained in an L-neighborhood of the geodesic connecting its endpoints.*

2.6 Nearest point projections and fellow traveling

Suppose that α is a subset of a Gromov hyperbolic space (X, d). The *nearest point projection* $p_\alpha\colon X \to \alpha$ sends each point $x \in X$ to a closest point to x in α. If α is Q-quasiconvex, then the nearest point projection is K-coarsely well defined, where K depends only on Q and the constant δ of hyperbolicity.

Suppose that α and β are two geodesics in X. We define the *K-fellow traveling set* for α with respect to β to be the subset of α contained in a K-neighborhood of β, i.e. $\alpha \cap N_K(\beta)$. If the diameter of the projection image $p_\alpha(\beta)$ is sufficiently large, then $p_\alpha(\beta)$ is contained in a bounded neighborhood of the geodesic β, and so is contained in a fellow traveling set.

In the special case that X is $\mathbb{H}^n$ and α is a geodesic, the closest point on α to any point $x \in X$ is unique, and so p_α is well-defined. Furthermore, for any geodesic β, the image $p_\alpha(\beta)$ is a subinterval of α, which we will refer to as the *nearest point projection interval*, or just the *projection interval*. Similarly, any K-fellow traveling set $\alpha \cap N_K(\beta)$ is also an interval, which we shall call the *K-fellow traveling interval*.

For $\mathbb{H}^n$, the hyperbolicity constant is $\delta = 2 \log 3$. We shall write δ_2 for the hyperbolicity constant for the pseudometric $d_{\widetilde{S}_h}$ on $\widetilde{S}_h$, and δ_3 for the hyperbolicity constant for the pseudometric $d_{\widetilde{S}_h \times \mathbb{R}}$ on $\widetilde{S}_h \times \mathbb{R}$. Both constants depend on the pseudo-Anosov f.

A standard consequence of δ-hyperbolicity is the useful property stated below that if the projection image of a geodesic β onto another geodesic α is large, then the two geodesics fellow travel and the projection image $p_\alpha(\beta)$ is contained in a bounded neighborhood of β.

Proposition 24. [KS24, Lemma 1.120] *Suppose that X is a δ-hyperbolic space and suppose that α and β are geodesics in X. If the diameter of the projection image $p_\alpha(\beta)$ is greater than 8δ, then the projection image is contained in a 6δ-neighborhood of β, i.e. $p_\alpha(\beta) \subseteq N_{6\delta}(\beta)$, and so $p_\alpha(\beta)$ is contained in the fellow traveling set $\alpha \cap N_{6\delta}(\beta)$.*

In $\mathbb{H}^2$, if two geodesics intersect at angle θ, then the size of the projection interval is roughly $\log(1/\theta)$. In fact,the same result holds for two geodesics that do not intersect, but are distance θ apart.

Proposition 25. *There is a constant $T_0 \geqslant 0$, such that for any unit speed geodesic γ_1 in $\mathbb{H}^2$ and any geodesic γ_2 such that*

- *γ_2 intersects γ_1 at the point $\gamma_1(0)$ at an angle $0 < \theta \leqslant \pi/2$, or*
- *the distance from γ_1 to γ_2 is $\theta > 0$, and the closest point occurs at $\gamma_1(0)$,*

then the nearest point projection interval $p_{\gamma_1}(\gamma_2)$ is equal to $\gamma_1([-T, T])$, where

$$\log \frac{1}{\theta} \leqslant T \leqslant \log \frac{1}{\theta} + T_0,$$

and furthermore, for all $|t| \leqslant \log \frac{1}{\theta}$, the distance from $\gamma_1(t)$ to γ_2 is at most $3/2$.

This is well known, we provide the details in [GMPU25, Appendix A]. In fact, for small $|t|$, the two geodesics are exponentially close, see [GMPU25, Appendix A] for further details.

We now record the useful fact that if two geodesics α and β intersect at angle θ, then the size of their projection intervals onto each other is roughly $\log \frac{1}{\theta}$, and furthermore, for any other geodesic γ, the overlap between the projection intervals for α and β on γ is bounded in terms of θ.

Proposition 26. *For any constant $\alpha_\Lambda > 0$ there is a constant $\rho_\Lambda > 0$ such that for any two geodesics in $\mathbb{H}^2$ which intersect at angle $\theta \geqslant \alpha_\Lambda$, and for any other geodesic γ, the intersection of the nearest point projection intervals of α and β to γ has diameter at most ρ_Λ.*

Proof. Suppose α and β intersect at the point p with angle $\theta \geqslant \alpha_\Lambda$. We shall choose $\rho_\Lambda = 2\log \frac{1}{\alpha_\Lambda} + 4T_0 + 16$. By Proposition 25, the radius of the nearest point projection interval of α to β, and also of β to α, is at most $\log \frac{1}{\alpha_\Lambda} + T_0$.

For any geodesic γ, let $I_\alpha = p_\gamma(\alpha)$ and $I_\beta = p_\gamma(\beta)$ be the nearest point projection intervals of α and β onto γ. Suppose that their overlap has size at least ρ_Λ, i.e. the length of $I_\alpha \cap I_\beta$ is at least ρ_Λ. If we truncate I_α and I_β by length T_0 at both ends, then the truncated intervals have overlap of length at least $\rho_\Lambda - 2T_0$. Let I be the interval of overlap for the truncated projection intervals, i.e. I is the closure of $I_\alpha \cap I_\beta \setminus N_{T_0}(\partial(I_\alpha \cap I_\beta))$.

By Proposition 25, each endpoint of I is distance at most $3/2$ from both α and β. We denote by a_1 and a_2 the points on α closest to each endpoint of I. It follows that the distance between a_1 and a_2 is at least $|I| - 2T_0 - 3$, and each point a_i is distance at most 3 from β.

We pick points b_i in β distance at most 3 from the points a_i. Then the nearest point projection of b_i to α is distance at most 6 from a_i. In particular, the diameter of the nearest point projection interval of β to α is at least $|I| - 2T_0 - 15 \leqslant \rho_\Lambda - 2T_0 - 15$. It follows from our choice of ρ_Λ that the diameter of the projection interval of β onto α is at least $2\log \frac{1}{\alpha_\Lambda} + 2T_0 + 1$, a contradiction. $\square$

Finally, we show that if two geodesics α and β have strictly nested projection intervals onto a third geodesic γ, i.e. $p_\gamma(\alpha) \subset p_\gamma(\beta)$, and α intersects γ, then α and β also intersect.

Proposition 27. *Let γ be a geodesic in $\mathbb{H}^2$ which intersects a geodesic ℓ_1, with projection interval $I_1 \subset \gamma$. Let ℓ_2 be a geodesic with projection interval $I_2 \subset \gamma$, such that $I_1 \subset I_2$. Then ℓ_1 and ℓ_2 intersect.*

Proof. Consider the nearest point projection map $p \colon \mathbb{H}^2 \to \gamma$. Consider the complement of the pre-image of I_1, i.e. $\mathbb{H}^2 \setminus p^{-1}(I_1)$. This has two connected components, which are separated by the geodesic ℓ_1. As I_1 is a strict subset of I_2, each endpoint of ℓ_2 is contained in a different complementary component, and so the endpoints of ℓ_2 are separated by ℓ_1, and so the two geodesics ℓ_1 and ℓ_2 intersect. $\square$

2.7 Lebesgue measures and the geodesic flow

We review the properties of the geodesic flow we will use, see for example [EW11].

A choice of basepoint x_0 in $\mathbb{H}^n$ determines a measure on the boundary sphere $\partial\mathbb{H}^n$, for example by choosing the disc or ball model for $\mathbb{H}^n$ with the basepoint as the center point, and giving $\partial\mathbb{H}^n$ the measure induced from the standard metric on the unit sphere. We will call this measure *Lebesgue measure*, and this measure depends on the choice of basepoint, though we will suppress this from our notation. Different choices of basepoint give measures which are absolutely continuous with respect to each other.

Let Γ be a discrete cocompact subgroup of isometries of $\mathbb{H}^n$, and let $M = \mathbb{H}^n/\Gamma$. Hopf [Hop71] showed that the geodesic flow g_t on the unit tangent bundle $T^1(M)$ is ergodic with respect to Liouville measure. As M is a compact hyperbolic manifold of constant negative curvature, Liouville measure is proportional to the Bowen-Margulis-Sullivan measure, the measure of maximal entropy for the geodesic flow. Liouville measure on $T^1(M)$ is the product of the measure determined by the hyperbolic metric on M, with Lebesgue measure on the unit tangent spheres. As the conditional measures on each unit tangent sphere determined by Liouville measure are Lebesgue measures, the measure on geodesics in M induced by Liouville measure is absolutely continuous with respect to the measure on geodesics in M induced by the product of Lebesgue measures on $(\partial\mathbb{H}^n \times \partial\mathbb{H}^n) \setminus \Delta$.

For the case of the Lebesgue surface measure, $n = 2$ and Γ is the fundamental group of the surface, $\pi_1 S$. For the case of the Lebesgue 3-manifold measure, $n = 3$, and Γ is the fundamental group of the mapping torus, $\pi_1 M$. In both cases, we will use the fact that a geodesic chosen according to the product of Lebesgue measures on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ is uniformly distributed in the unit tangent bundle of the compact quotient manifold, almost surely. We will use the following version of this result.

Proposition 28. *Let Γ be a discrete cocompact group of isometries of $\mathbb{H}^n$, and let B be a Borel set in $M = \mathbb{H}^n/\Gamma$. Then for almost all geodesics γ in $T^1(M)$,*

$$\lim_{T\to\infty} \frac{1}{T} |\gamma([0,T]) \cap B| = \mathbf{vol}(B).$$

2.8 Random walks and hitting measures

We recall some results in the theory of random walks on countable groups acting on Gromov hyperbolic spaces. Suppose that a countable group G acts on a δ-hyperbolic space X. The results we state do not require X to be locally compact or the action to be locally finite. But for our purposes, $G \curvearrowright X$ will always be either $\pi_1(S) \curvearrowright \widetilde{S}_h$ or $\pi_1(M) \curvearrowright \widetilde{S}_h \times \mathbb{R}$, which are cocompact actions on locally compact spaces. We say the action of a group G on a space X is *nonelementary* if it contains two independent loxodromic isometries of X. We say a probability measure μ on G is *geometric* if it has finite support and the semigroup generated by its support is equal to G.

A random walk of length n on G is a random product $w_n = g_1 \cdots g_n$ where each g_i is chosen independently according to a probability measure μ on G. We call the elements g_i the *steps* of the random walk. Passing to infinitely many steps, we may consider the sequence of steps (g_n) to be an element of $(G, \mu)^{\mathbb{Z}}$. We call $(G, \mu)^{\mathbb{Z}}$ the *step space*. The *location* w_n at time n is determined by $w_0 = 1 \in G$, and $w_{n+1} = w_n g_{n+1}$ for all $n \in \mathbb{Z}$. The *location space* is the probability space $(G^{\mathbb{Z}}, \mathbb{P})$, where $\mathbb{P}$ is the pushforward of the product measure under the map that sends (g_n) to (w_n). A choice of basepoint $x_0 \in X$ gives a sequence $(w_n x_0)$ which we shall also call a *sample path* of the random walk.

Theorem 29. [Kai94][MT18] *Suppose that G is a countable group that has a nonelementary action on a Gromov hyperbolic space X. Suppose that μ is a nonelementary probability measure on G, that is, the support of μ generates a nonelementary subgroup of G for its action on X. Then almost all (bi-infinite) sample paths $w = (w_n)_{n \in \mathbb{Z}}$ for the μ-random walk on G, the sequences $(w_n x_0)$ converge as $n \to \infty$ and $n \to -\infty$ to the Gromov boundary ∂X of X. The convergence defines a pair of non-atomic measures ν and $\check{\nu}$ on ∂X.*

We call the measures on ∂X obtained in Theorem 29 the *hitting measures* for the random walk. If μ is symmetric, then the forward and backward hitting measures are equal, i.e. $\nu = \check{\nu}$.

For almost every sample path $w = (w_n)_{n \in \mathbb{Z}}$, the forward and backward limits $x_\infty^+ = \lim_{n \to \infty} w_n x_0$ and $x_\infty^- = \lim_{n \to -\infty} w_n x_0$ are different from each other. Hence, almost every bi-infinite sample path w defines a bi-infinite geodesic γ_w in X and all choices for γ uniformly fellow travel, in the sense that the fellow traveling constant depends only on the Gromov-hyperbolicity constant. Making a choice for the geodesic, we call it the geodesic *tracked* by the sample path.

The distance $d_X(x_0, w_n x_0)$ grows linearly with high probability. This was shown by [BMSS23] for μ with finite exponential moment, and by Gouëzel [Gou22, Theorem 1.1] for general μ.

Lemma 30. [BMSS23][Gou22] *Suppose that G is a countable group with a nonelementary action on a Gromov hyperbolic space X. Suppose that μ is a nonelementary probability measure on G. Then there are constants $\ell > 0, K \geqslant 0$ and $0 < c < 1$ such that*

$$\mathbb{P}(d_X(x_0, w_n x_0) \leqslant \ell n) \leqslant K c^n.$$

Furthermore, if μ has finite exponential moment in X, then for any $\epsilon > 0$ there are constants $\ell > 0, K \geqslant 0$ and $c < 1$ such that

$$\mathbb{P}(|\tfrac{1}{n} d_X(x_0, w_n x_0) - \ell| \geqslant \epsilon) \leqslant K c^n.$$

The Gromov product of three points a, b, c in a metric space X is defined to be

$$(b \, . \, c)_a = \tfrac{1}{2}(d_X(a, b) + d_X(a, c) - d_X(b, c))$$

When X is δ-hyperbolic, the Gromov product $(b \,.\, c)_a$ equals the distance from a to a geodesic from b to c, up to a bounded error depending only on δ. In this case, the product can be extended to points which lie in the boundary, i.e. we may choose $b, c \in \overline{X} = X \cup \partial X$.

The "shadow" will, roughly speaking, be the set of all points c in $\overline{X}$ such that any geodesic from a to c passes close to b. Formally:

Definition 31. Suppose that a is a point in a Gromov hyperbolic space X and b is a point in $\overline{X}$. Suppose that $r \geqslant 0$ is a constant. The *shadow* $\mho_a(b, r) \subseteq \overline{X}$ consists of the closure of the set of all points c, such that $(b \,.\, c)_a \geqslant r$.

This differs from the usual definition of shadows, which is the closure of the following set, where again $a \in X$ and $b \in \overline{X}$:

$$\mho_a(b, r) = \{c \in \overline{X} \mid (b \,.\, c)_a \geqslant d_X(a, b) - r\}.$$

The two definitions are equivalent by setting $r = d_X(a, b) - R$ for points $b \in X$, but our definition extends more conveniently to points $b \in \partial X$.

We will use the following Gromov product estimates from [BMSS23]. We state the version incorporating the linear progress results of [Gou22]. As shadows are defined in terms of the Gromov product, we also state the results in terms of shadows.

Proposition 32. [BMSS23, Proposition 2.11][Gou22, Theorem 1.1] *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov-hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G with a finite exponential moment. Then there are constants $K > 0$ and $c < 1$ such that for all $0 \leqslant i \leqslant n$ and all $R > 0$ one has*

$$\mathbb{P}\left((x_0 \,.\, w_n x_0)_{w_i x_0} \geqslant R\right) \leqslant K c^R.$$

Furthermore, using the definition of shadows,

$$\mathbb{P}\left(\nu(\mho_{x_0}(w_n x_0, R))\right) \leqslant K c^R.$$

Let $\gamma(t_n)$ be a closest point on γ to $w_n x_0$. Then combining Lemma 30 and Proposition 32 gives the following linear progress result for the t_n.

The above results imply that the distance from a location $w_n x_0$ to the tracked geodesic γ is given by a probability measure with exponential decay.

Proposition 33. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G. Then there are constants $K \geqslant 0$ and $c < 1$ such that $\mathbb{P}(d_X(w_n x_0, \gamma) \geqslant r) \leqslant K c^r$.* □

The Borel-Cantelli Lemma then gives the following corollary.

Corollary 34. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G with finite exponential moment with respect to X. Then there is a constant $D > 0$, such that for almost all bi-infinite sample paths $w = (w_n)$, there is a tracked geodesic $\gamma = \gamma_w$, and an integer N such that for all $n \geqslant N$*

$$d_X(w_n x_0, \gamma) \leqslant D \log n.$$

Proof. By Proposition 33, for any $D > 0$ we have

$$\mathbb{P}(d_X(w_n x_0, \gamma) \geqslant D \log n) \leqslant K \exp(D \log n \log c).$$

We choose $D > 0$ such that $D \log c < -1$. Set $\alpha = -D \log c$. Then

$$\sum_n \mathbb{P}(d_X(w_n x_0, \gamma) \geqslant D \log n) \leqslant K \sum_n \frac{1}{n^\alpha} < \infty.$$

Hence, by the Borel-Cantelli Lemma, we deduce that for almost every sample path, there is a natural number N (that depends on the path) such that $d(w_n x_0, \gamma) < D \log n$ for all $n \geqslant N$. $\square$

Corollary 35. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G. Then there are constants $\ell > 0, K \geqslant 0$ and $0 < c < 1$ such that*

$$\mathbb{P}(t_n \leqslant \ell n) \leqslant K c^n.$$

Furthermore, if μ has finite exponential moment with respect to X, then for any $\epsilon > 0$ there are constants $\ell > 0, K \geqslant 0$ and $c < 1$ such that

$$\mathbb{P}(|\tfrac{1}{n} t_n - \ell| \geqslant \epsilon) \leqslant K c^n.$$

We now use the Borel-Cantelli Lemma to show that the gap between t_n and t_{n+1} is at most $\log n$.

Proposition 36. *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a nonelementary probability measure on G with finite exponential moment with respect to X. Then there is a constant $D > 0$, such that for almost all sample paths, there is a tracked geodesic γ, and an integer N such that for all $n \geqslant N$*

$$|t_{n+1} - t_n| \leqslant D \log n.$$

Proof. By Proposition 32 there are constants $K_1 > 0$ and $c_1 < 1$ such that $\mathbb{P}(d_X(w_n x_0, w_{n+1} x_0) \geqslant R) \leqslant K_1 c_1^R$. By Proposition 33, there are constants $K_2 > 0$ and $c_2 < 1$ such that for all n, we have that $\mathbb{P}(d_X(w_n x_0, \gamma) \geqslant R) \leqslant K_2 c_2^R$. By the triangle inequality,

$$|t_{n+1} - t_n| \leqslant d_X(\gamma(t_n), w_n x_0) + d_X(w_n x_0, w_{n+1} x_0) + d_X(w_{n+1} x_0, \gamma(t_{n+1})).$$

If $|t_{n+1} - t_n| \geqslant 3R$, then at least one of the terms on the right is at least R. Therefore, $\mathbb{P}(|t_{n+1} - t_n| \geqslant 3R) \leqslant \max\{K_1 c_1^R, K_2 c_2^R\}$. The result then follows from the Borel-Cantelli Lemma applied the same way as in the proof of Corollary 34. □

Finally, we verify that the $(\nu \times \check{\nu})$-measure, as defined in Theorem 29, of the diagonal $\Delta \subset \partial X \times \partial X$ is zero. We may define a neighborhood of the diagonal as follows (using any basepoint $a \in X$).

$$N_r(\Delta) = \bigcup_{b \in \partial X} \mho_a(b, r) \times \mho_a(b, r)$$

Lemma 37. [Mah12, Proposition 4.7] *Suppose that G is a countable group with a nonelementary action on a geodesic Gromov hyperbolic space X with basepoint x_0. Suppose that μ is a geometric probability measure on G. Then there are constants $K \geqslant 0$ and $c < 1$ such that $\nu \times \check{\nu}(N_r(\Delta)) \leqslant Kc^r$, with $\nu \times \check{\nu}$ again as in Theorem 29.*

Proof. By the action of G on $(\partial X \times \partial X, \nu \times \check{\nu})$, it suffices to show this for $d_X(x_0, \gamma)$. Suppose that $d_X(x_0, \gamma) \geqslant r$. As the Gromov product of three points $(b \,.\, c)_a$ is, up to an error of at most 2δ, equal to the distance from a to a geodesic from b to c, the endpoints of γ are contained in the neighborhood $N_{r+2\delta}(\Delta)$. The probability that this occurs is at most $\nu \times \check{\nu}(N_{r+2\delta}(\Delta)) \leqslant Kc^{r+2\delta}$, as required. □

2.9 The pseudo-metric on the surface and the flat metric

In this section we review some well known results that relate hyperbolic and flat metrics on surfaces, see for example [Kap01, Chapter 11].

A flat structure S_q on a closed surface S is a Euclidean cone metric with finitely many cone points, each of whose angles are integer multiples of π. Furthermore, for any sufficiently small coordinate chart disjoint from the cone points, there is a preferred choice of orthogonal directions, known as either the horizontal and vertical directions, or alternatively the real and imaginary directions. The integral lines of the horizontal directions give a (singular) foliation called the horizontal foliation. Similarly, the integral lines of the vertical directions give a (singular) foliation called the vertical foliation. Flat lengths of orthogonal arcs define transverse measures for the foliations.

The Cannon-Thurston metric given by Equation (1) is $\pi_1(S)$-invariant. Hence, the restriction of the infinitesimal pseudometric on $\widetilde{S}_h \times \mathbb{R}$ to S_0 gives rise to an infinitesimal pseudometric on S_h, which we have denoted by $d_{\widetilde{S}_h}$. By identifying points on S_h that are pseudometric distance zero apart, i.e. points which lie in the same component of $S_h \setminus (\Lambda_+ \cup \Lambda_-)$, we get a quotient of S_h which is isometric to a flat metric S_q on S. The quotient map sends the invariant laminations (Λ_-, dy) and (Λ_+, dx) to the real and imaginary measured foliations ($\mathcal{F}_r$ and $\mathcal{F}_i$ respectively) for the flat metric S_q. As both (pseudo-)metrics are defined on the closed surface S, they give quasi-isometric metrics on the universal cover. We record this statement as a proposition to fix notation.

Proposition 38. *Suppose that (S_h, Λ) is a hyperbolic metric on S together with a full pair of measured laminations. Then the hyperbolic metric $d_{\mathbb{H}^2}$ and the Cannon-Thurston pseudo-metric $d_{\widetilde{S}_h}$ on the universal cover $\widetilde{S}_h$ are quasi-isometric, i.e. there are constants $Q_\Lambda \geqslant 1$ and $c_\Lambda \geqslant 0$ such that for any points p and p' in the universal cover,*

$$\frac{1}{Q_\Lambda} d_{\widetilde{S}_h}(p, p') - c_\Lambda \leqslant d_{\mathbb{H}^2}(p, p') \leqslant Q_\Lambda d_{\widetilde{S}_h}(p, p') + c_\Lambda.$$

By compactness of the circle direction or equivalently $\mathbb{Z}$-periodicity by the action of the pseudo-Anosov f, the constants in Proposition 38 remain uniform over any choice S_z as fiber. More generally, if a doubly degenerate surface group in $\mathrm{PSL}(2, \mathbb{C})$ has bounded geometry then the constants in Proposition 38 will remain uniform for the quasi-isometry between the pseudo-metric and the flat metric for any S_z.

3 Singularity of measures

In this section, we prove the singularity of measures, namely Theorem 4, by showing that typical geodesics have different behavior for the pushforwards of the surface measures (Theorem 6) than for the 3-manifold measures (Theorem 8). We show that for almost all geodesics in $\widetilde{S}_h$ with respect to the surface measures, the geodesics they determine in $\widetilde{S}_h \times \mathbb{R}$ spend a positive proportion of time close to the base fiber. On the other hand, for almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$ with respect to the 3-manifold measures, the proportion of time spent close to the base fiber tends to zero. We prove these facts for Lebesgue measures in Section 3.2 and for hitting measures for random walks in Section 3.3. We start by recording some useful facts about geodesics which we will use in the subsequent sections.

Suppose that γ is an oriented geodesic in $\widetilde{S}_h$. We denote its pair of limit points in $\partial \widetilde{S}_h$ by γ_+ and γ_-.

Definition 39. We say that a bi-infinite geodesic γ in $\widetilde{S}_h$ is *non-exceptional* if

- its limit points γ_- and γ_+ are distinct from the limit points of any boundary leaf of any ideal complementary region of either of the invariant laminations, and
- the limit points have distinct images under the Cannon-Thurston map, that is $\iota(\gamma_-) \neq \iota(\gamma_+)$.

As we see below, for surface measures, almost all geodesics in $\widetilde{S}_h$ are non-exceptional.

Proposition 40. *Suppose that $f \colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Let ν be one of the surface measures from Definition 1 or Definition 2. Then ν-almost all geodesics in $\widetilde{S}_h$ are non-exceptional.*

Proof. Let γ be a geodesic in $\widetilde{S}_h$. Suppose that its image $\iota(\gamma)$ has a single limit point at infinity. Then γ is either a leaf of an invariant lamination, or contained in an ideal complementary region. Being ideal polygons with finitely many sides, there are only countably many ideal complementary regions. So the collection of geodesics contained in ideal complementary regions has measure zero as the measures are all non-atomic.

So we may consider leaves of invariant laminations. For Lebesgue measure, Birman and Series [BS85, Theorem II] showed that the collection of endpoints of all simple geodesics has measure zero in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. For hitting measure, this result follows from double ergodicity of the action of $\pi_1(S)$ on the boundary, due to Kaimanovich [Kai03, Theorem 17]. For completeness, we give the details for hitting measure below.

Let Δ denote the diagonal in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Suppose Λ is a geodesic lamination and $\partial\Lambda \subset \partial\widetilde{S}_h \times \partial\widetilde{S}_h \setminus \Delta$ be the endpoints of leaves of Λ. The action of the fundamental group $\pi_1(S)$ on $\partial\widetilde{S}_h \times \partial\widetilde{S}_h \setminus \Delta$ is ergodic for the hitting measure. Since Λ is $\pi_1(S)$-invariant, the set $\partial\Lambda$ has measure 0 or 1. Suppose ℓ is a leaf of Λ. Suppose that γ is a geodesic that crosses ℓ. Then γ does not lie in Λ. We can then choose small neighborhoods U_+ and U_- of γ_+ and γ_- such that any geodesic with one endpoint in U_+ and the other endpoint in U_- also crosses ℓ. Thus, such a geodesic does not lie in Λ. As open sets have positive measure, $\partial\Lambda$ has measure strictly less than 1. Hence, $\partial\Lambda$ has measure zero as required. □

3.1 Quasigeodesics from loxodromics

Both S and M are compact, and their fundamental groups are torsion free. The hyperbolic metrics on S and M give maps $\rho_S \colon \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^2)$ and $\rho_M \colon \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$, whose images are torsion-free cocompact lattices. In particular, in both cases, every non-trivial element of the fundamental group maps to a loxodromic isometry. We now show that the image of an axis for a non-trivial element of $\pi_1(S)$ is a quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$.

Proposition 41. *Suppose that $f \colon S \to S$ is a pseudo-Anosov map, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations, and let g be a non-trivial element of $\pi_1(S)$ with axis α in $\widetilde{S}_h$. Then there are constants $Q_\alpha \geqslant 1, c_\alpha$ and K_α such that the image of the axis $\iota(\alpha)$ in $\widetilde{S}_h \times \mathbb{R}$ is (Q_α, c_α)-quasigeodesic. In particular, the geodesic $\overline{\alpha}$ connecting the endpoints of $\iota(\alpha)$ is contained in a K_α-neighborhood of S_0.*

Proof. The isometry $\rho_S(g) \in \mathrm{Isom}(\mathbb{H}^2)$ is loxodromic. Let α be the axis of $\rho_S(g)$ in $\mathbb{H}^2$. By abuse of notation, we will also write g for the image of g in $\pi_1(M)$ under the (injective) inclusion map $i \colon \pi_1(S) \to \pi_1(M)$. Then $\rho_M(g) \in \mathrm{Isom}(\mathbb{H}^3)$ is also loxodromic. As $\widetilde{S}_h \times \mathbb{R}$ is quasi-isometric to $\mathbb{H}^3$, g also acts loxodromically on $\widetilde{S}_h \times \mathbb{R}$. Recall that $\iota \colon \widetilde{S}_h \to \widetilde{S}_h \times \mathbb{R}$ is the inclusion map. The image $\iota(\alpha)$ is an $\rho_M(g)$-invariant path in $\widetilde{S}_h \times \mathbb{R}$. Hence, for constants (Q_α, c_α) that depend on α, the path $\iota(\alpha)$ is a (Q_α, c_α)-quasigeodesic. We shall write $\overline{\alpha}$ for the geodesic in $\widetilde{S}_h \times \mathbb{R}$ connecting the endpoints of $\iota(\alpha)$. It follows that $\overline{\alpha}$ is the axis for g acting on $\widetilde{S}_h \times \mathbb{R}$. By the Morse lemma, Lemma 23, there is a constant $K_\alpha > 0$ such that

$\iota(\alpha)$ is contained in an K_α-neighborhood of $\overline{\alpha}$. As $\iota(\alpha)$ is contained in S_0, the proposition follows. $\square$

We next show that there is an upper bound P_α on the length of the nearest point projection interval $p_\alpha(\ell)$, for any leaf ℓ of an invariant lamination.

Proposition 42. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Let g be a non-trivial element of $\pi_1(S)$ with axis α in $\widetilde{S}_h$. Then there is a constant $P_\alpha > 0$ such that for any leaf ℓ of either of the invariant laminations, the nearest point projection of ℓ to α and α to ℓ has length at most P_α.*

Proof. Every closed geodesic in S_0 intersects both invariant laminations. Hence, any segment of α with length L_α intersects both laminations Λ_+ and Λ_-. By compactness, there is also a minimum angle $\theta_\alpha > 0$ for an intersection of α with any leaf of an invariant lamination. The above two properties together with Proposition 25 imply that there is an upper bound P_α for the length of the nearest point projection interval $p_\alpha(\ell)$, for any leaf ℓ of an invariant lamination, and similarly for $p_\ell(\alpha)$. $\square$

The axis α projects to a closed geodesic in S_0. Let L_α be the length of this geodesic. Thus, L_α is the translation length of g on $\widetilde{S}_h$. Note that we may replace P_α by a larger constant to assume that $P_\alpha \geqslant L_\alpha$, and it will be convenient to do so. We now show that the limit points of $\iota(\alpha)$ are disjoint from the limit sets of any ladder $F(\ell)$ of any leaf ℓ of an invariant lamination.

Proposition 43. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that g is a non-trivial element of $\pi_1(S)$. For any leaf ℓ of an invariant lamination, the fixed points $\overline{\alpha}_+$ and $\overline{\alpha}_-$ of $\rho_M(g)$ are disjoint from the limit points of $F(\ell)$ in $\partial(\widetilde{S}_h \times \mathbb{R})$.*

Proof. For an ideal complementary region C of an invariant lamination, we denote by $s(C)$ the number of sides of C. Since there are finitely many ideal complementary regions, the numbers $s(C)$ over all C has a maximum which we denote by $s_{\max}$.

Let ℓ be a leaf of an invariant lamination. Breaking symmetry, suppose that ℓ lies in Λ_+. By Proposition 42, the projection interval $p_\alpha(\ell)$ has length at most P_α. As $\rho_S(g)$ acts on α by translation, we may choose n large enough such that the projection interval $p_\alpha(\rho_S(g)^n\ell)$ is distance greater than $s_{\max}P_\alpha$ from the projection interval $p_\alpha(\ell)$.

Suppose that ℓ and $\rho_S(g)^n\ell$ intersect a common ideal complementary region of the other lamination Λ_-. Traversing in a cyclic order, we may choose a cyclic sequence of boundary geodesics $\ell'_1, \cdots, \ell'_j$ such that ℓ'_1 intersects ℓ and ℓ'_j intersects $\rho_S(g)^n\ell$. But then the projection intervals $p_\alpha(\ell)$ and $p_\alpha(\rho_S(g)^n\ell)$ are at most $jP_\alpha < s_{\max}P_\alpha$ distance apart, a contradiction. We deduce that ℓ and $\rho_S(g)^n\ell$ do not intersect a common ideal complementary region of Λ_-.

On the other hand, since $P_\alpha > L_\alpha$, there is a leaf of Λ_+ separating ℓ and $\rho_S(g)^n\ell$.

By Proposition 19 and Proposition 20, the distance between $F(\ell)$ and $F(\rho_S(g)^n\ell)$ is at least $\epsilon > 0$. By Proposition 21, their limit sets are disjoint. The ladder $F(\rho_S(g)^n\ell)$ equals $\rho_M(g)^n F(\ell)$ and thus the limit set of $F(\ell)$ cannot contain any fixed points of $\rho_M(g)$, as required. $\square$

In a similar vein, we now show that there is an upper bound $P_{\overline{\alpha}}$ on the nearest point projection in $\widetilde{S}_h \times \mathbb{R}$ of the axis $\overline{\alpha}$ to any ladder $F(\ell)$ over any leaf ℓ of an invariant lamination.

Corollary 44. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that g is a non-trivial element of $\pi_1(S)$ with axis $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. Then there is a constant $P_{\overline{\alpha}} > 0$ such that for any leaf ℓ of an invariant lamination, the diameter of the projection image $p_{\overline{\alpha}}(F(\ell))$ is at most $P_{\overline{\alpha}}$.*

Proof. Suppose that there is a sequence of leaves ℓ_n of the invariant laminations such that the diameters of the projections $p_{\overline{\alpha}}(F(\ell_n))$ tend to infinity as $n \to \infty$. By Proposition 41, the image $\iota(\alpha)$ is contained in a bounded neighborhood of $\overline{\alpha}$. Hence, the diameters of the images of the ladders $F(\ell_n)$ under the nearest point projection to $\iota(\alpha)$, also tend to infinity. As the ladders $F(\ell_n)$ are quasiconvex, the diameters of the subsets of $\iota(\alpha)$ contained in a bounded neighborhood of $F(\ell_n)$ tends to infinity. Hence, the diameters of the nearest point projections of $\ell_n \times \{0\}$ to $\iota(\alpha)$ tends to infinity, contradicting Proposition 42 which states that the diameters are bounded above by P_α. $\square$

We now show that if two points x and y in $\widetilde{S}_h$ have nearest point projections to α which are far apart, then their images $\iota(x)$ and $\iota(y)$ in $\widetilde{S}_h \times \mathbb{R}$ have nearest point projections to $\overline{\alpha}$ which are far apart. As the inclusion map distorts distances, *a priori*, the inclusion of the nearest point projection of x to α need not be close to the nearest point projection of $\iota(x)$ to $\overline{\alpha}$.

Proposition 45. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that g is a non-trivial element of $\pi_1(S)$, with axis α in $\widetilde{S}_h$, and axis $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. Then there are constants $Q \geqslant 1$ and $c \geqslant 0$ such that for any two points x and y in $\widetilde{S}_h \cup \partial\widetilde{S}_h$,*

$$d_{\widetilde{S}_h \times \mathbb{R}}(p_{\overline{\alpha}}(\iota(x)), p_{\overline{\alpha}}(\iota(y))) \geqslant \frac{1}{Q} d_{\widetilde{S}_h}(p_\alpha(x), p_\alpha(y)) - c.$$

Proof. Let $I = [p, q]$ be the subinterval of α with endpoints $p := p_\alpha(x)$ and $q := p_\alpha(y)$. We may assume that $d_{\widetilde{S}_h}(p, q) \geqslant 2L_1$, where $L_1 = L_\alpha + 2P_\alpha$, where L_α is the translation length of g, and P_α is the largest size of the projection of any leaf of an invariant lamination to α, from Proposition 42. We shall choose $Q = Q_\alpha$ and $c = 2L_1/Q_\alpha + c_\alpha + 2K_\alpha + 2P_{\overline{\alpha}}$, where Q_α and c_α are the quasigeodesic constants for $\iota(\alpha)$ in $\widetilde{S}_h \times \mathbb{R}$ from Proposition 41, and K_α is the corresponding Morse constant.

Let β be a segment of I of length L_1. The central segment of β of length L_α intersects leaves of both laminations. Suppose that ℓ is such a leaf. By our choice of L_1, the projection interval $p_\alpha(\ell)$ is contained in the interior of β, which in turn is contained in I. Let γ be the geodesic spanned by x and y. By Proposition 27, as $p_\alpha(\ell) \subset p_\alpha(\gamma)$, and as α and ℓ intersect, ℓ and γ also intersect.

Let β_1 be an initial segment of I of length L_1, and let β_2 be a terminal segment of I of length L_1. Since $I|I| \geqslant 2L_1$, the segments β_1 and β_2 are disjoint. By the conclusions of the previous paragraph, there exists leaves ℓ_1 and ℓ_2 of an invariant lamination such that

- ℓ_1 intersects β_1 and γ and $p_\alpha(\ell_1)$ lies in the interior of β_1, and
- ℓ_2 intersects β_2 and γ and $p_\alpha(\ell_2)$ lies in the interior of β_2.

It follows that ℓ_1 and ℓ_2 are disjoint and divide $\widetilde{S}_h$ into three regions A_1, A_2 and A_3 such that

- A_1 is adjacent to ℓ_1 and contains x,
- A_2 lies between ℓ_1 and ℓ_2, and
- A_3 is adjacent to ℓ_2 and contains y.

The leaves ℓ_1 and ℓ_2 divide α into arcs α_1, α_2, and α_3 such that α_i is contained in A_i. Furthermore, $p_\alpha(x)$ is contained in α_1, and $p_\alpha(y)$ is contained in α_3. The arc $I \setminus (\beta_1 \cup \beta_2)$ is contained in α_2 and its length is $d_{\widetilde{S}_h}(p, q) - 2L_1$. It follows that the length of α_2 is at least $d_{\widetilde{S}_h}(p, q) - 2L_1$.

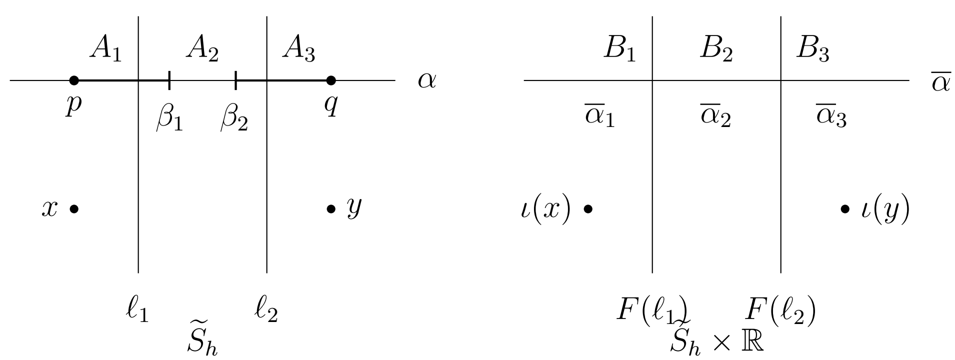

Figure 4: Notation for the complements of the leaves ℓ_i and ladders $F(\ell_i)$.

The ladders $F(\ell_1)$ and $F(\ell_2)$ are convex in $\widetilde{S}_h \times \mathbb{R}$. Since ℓ_1 and ℓ_2 are disjoint, $F(\ell_1)$ and $F(\ell_2)$ are also disjoint. Thus they similarly divide $\widetilde{S}_h \times \mathbb{R}$ into three regions B_1, B_2 and B_3 where

- B_1 is adjacent to $F(\ell_1)$,
- B_2 is between $F(\ell_1)$ and $F(\ell_2)$, and

- B_3 is adjacent to $F(\ell_2)$.

Note that $\iota(A_i) \subseteq B_i$. This implies that the initial limit point of $\overline{\alpha}$ is contained in the limit set of B_1, and the terminal limit point of $\overline{\alpha}$ is contained in the limit set of B_3. By Proposition 43, the limit points of $\overline{\alpha}$ are not contained in the limit points of either $F(\ell_1)$ or $F(\ell_2)$, so the ladders $F(\ell_i)$ also divide $\overline{\alpha}$ into three components, $\overline{\alpha}_1$, $\overline{\alpha}_2$ and $\overline{\alpha}_3$ so that $\overline{\alpha}_i \subset B_i$. Since $\iota(x)$ lies in B_1 its nearest point projection to $\overline{\alpha}$ is contained in the nearest point projection of B_1 to $\overline{\alpha}$. Since the boundary of B_1 is the ladder $F(\ell_1)$, which is convex, we deduce that

$$p_{\overline{\alpha}}(\iota(x)) \subseteq \overline{\alpha}_1 \cup p_{\overline{\alpha}}(F(\ell_1)).$$

In particular, the projection of $\iota(x)$ to $\overline{\alpha}$ is contained in a $P_{\overline{\alpha}}$-neighborhood of α_1. Similarly, the projection of $\iota(y)$ to $\overline{\alpha}$ is contained in $\overline{\alpha}_3 \cup p_{\overline{\alpha}}(F(\ell_2))$, and so is contained in a $P_{\overline{\alpha}}$-neighborhood of $\overline{\alpha}_3$.

The intersection points of ℓ_1 and ℓ_2 with α are endpoints of α_2, whose length is at least $|I| - 2L_1$. Since $\iota(\alpha)$ is a (Q_α, c_α)-quasigeodesic in $\widetilde{S_h} \times \mathbb{R}$, the distance between the intersection points of $F(\ell_1)$ and $F(\ell_2)$ with $\iota(\alpha)$ is at least

$$\frac{1}{Q_\alpha}(|I| - 2L_1) - c_\alpha.$$

Being a quasigeodesic, $\iota(\alpha)$ is contained in an K_α-neighborhood of $\overline{\alpha}$. Therefore, the distance between the nearest point projections of $F(\ell_1)$ and $F(\ell_2)$ to $\overline{\alpha}$ is at least

$$\frac{1}{Q_\alpha}(\mathrm{diam}(|I| - 2L_1) - c_\alpha - 2K_\alpha.$$

By Corollary 44, the diameter of $p_{\overline{\alpha}}(F(\ell_1))$, and similarly $p_{\overline{\alpha}}(F(\ell_2))$, is at most $P_{\overline{\alpha}}$. It follows that the distance in $\widetilde{S_h} \times \mathbb{R}$ between $p_{\overline{\alpha}}(F(\ell_1))$ and $p_{\overline{\alpha}}(F(\ell_2))$ is at least

$$\frac{1}{Q_\alpha}(|I| - 2L_1) - c_\alpha - 2K_\alpha - 2P_{\overline{\alpha}} = \frac{1}{Q} d_{\widetilde{S}_h}(p, q) - c,$$

where $Q = Q_\alpha$ and $c = 2L_1/Q_\alpha + c_\alpha + 2K_\alpha + 2P_{\overline{\alpha}}$. Therefore, the distance between the projections of $\iota(x)$ and $\iota(y)$ to $\overline{\alpha}$ is at least $\frac{1}{Q}\mathrm{diam}(p_\alpha(\gamma)) - c$, where the constants Q and c depend only on α and not on x and y, as required. □

We will also use the well known fact that for a discrete group of isometries of $\mathbb{H}^n$, for any loxodromic element g with axis α, the size of the projections of all of the translates of the axis α to α is bounded, see for example [BBF15, Example 2.1(1)].

Proposition 46. *Suppose that G is a countable group acting locally finitely by loxodromics on $\mathbb{H}^n$. Suppose that g is a loxodromic element with geodesic axis α. Then there is a constant L such that for any distinct translate $h\alpha \neq \alpha$ of the axis, the size of the projection interval $p_\alpha(h\alpha)$ is at most L.*

3.2 Lebesgue measure

In this section we will consider geodesics chosen using Lebesgue measure on either $\partial\widetilde{S}_h$ or $\partial(\widetilde{S}_h \times \mathbb{R})$. We start with the surface case, and show that for the pushforward of Lebesgue measure on $\partial\widetilde{S}_h$, almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$ spend a positive proportion of time close to the base fiber S_0. This shows the Lebesgue measure case of Theorem 6.

Lemma 47. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that ν is the pushforward of Lebesgue measure on $\partial\widetilde{S}_h$ under the Cannon-Thurston map. Then there are constants $R \geqslant 0$ and $\epsilon > 0$ such that for $(\nu \times \nu)$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$,*

$$\lim_{T\to\infty} \frac{1}{T}|\{t \in [0,T] \mid \overline{\gamma}(t) \in N_R(S_0)\}| \geqslant \epsilon.$$

We give a brief overview of the argument. Let α be a geodesic in $\widetilde{S}_h$ which covers a closed geodesic β in S_h. By the results of the previous section, if a geodesic γ in $\widetilde{S}_h$ has a large fellow travel with α, then $\overline{\gamma}$ has a large fellow travel with $\overline{\alpha}$. By ergodicity, a typical geodesic γ in S_h spends a positive proportion of time fellow-traveling any closed geodesic β in S_h. This implies that $\overline{\gamma}$ spends a positive proportion of time close to S_0. We now give a precise version of this argument.

Proof. Let α be a geodesic in $\widetilde{S}_h$ which projects to a closed geodesic β in S_h. By abuse of notation, we denote the lift of β to $T^1(S_h)$ also by β. Let δ_1 be a sufficiently small constant such that $N_{\delta_1}(\beta)$ is a regular neighborhood of β in $T^1(S_h)$. In particular, this implies that δ_1 is less than the injectivity radius of $T^1(S_h)$. We fix a constant $L > 0$ such that

$$L \geqslant Qc + 11\delta_3 + 1, \tag{2}$$

where Q and c are constants in Proposition 45, and δ_3 is the constant of hyperbolicity for $\widetilde{S}_h \times \mathbb{R}$.

Let η be an oriented geodesic in $\widetilde{S}_h$ which intersects $N_{\delta_1}(\alpha)$ in a segment of length L, and let v be the tangent vector to η at the first point of intersection between η and $N_{\delta_1}(\alpha)$. By abuse of notation, we shall also write v for its image in the quotient $T^1(S_h)$.

Let $A(r)$ be the disc of radius r centered at v, perpendicular to the lift of η in $T^1(S_h)$. For any constant $\epsilon > 0$ there is a constant δ such that the forward image of any point w in $A(r)$ under the geodesic flow intersects $N_{\delta_1}(\beta)$ in a segment of length L, up to additive error at most ϵ, and furthermore w intersects $N_{\delta_1}(\beta)$ within distance ϵ under the geodesic flow. Let δ_2 be the value of δ corresponding to $\epsilon_2 = \min\{\delta_1, \delta_3, L/2\}$. We can replace δ_2 by any smaller positive number, so in particular we may assume $\delta_2 \leqslant \delta_3$.

Now choose V to be the image of $A(\delta_2)$ under the forward and backward geodesic flows of distance $\frac{1}{2}\delta_2$. The interior of V is an open regular neighborhood of v. Every geodesic flow line that intersects V intersects it in a segment of length δ_2, and intersects $N_\delta(\beta)$ in a segment of length at least $L - \epsilon_2$, and at most $L + \epsilon_2$, starting within distance ϵ_2 of the intersection of the flow line with V. As $N_{\delta_2}(\beta)$ is a regular neighborhood of β, for any flow line γ, segments of $\gamma \cap N_{\delta_2}(\beta)$ corresponding to distinct intersections with V do not intersect.

As the geodesic flow is ergodic, almost all geodesic flow lines spend a positive proportion of time in V, i.e.

$$\lim_{T\to\infty} \frac{1}{T} |\gamma(0,T) \cap V| = \mathbf{vol}_{T^1(S_h)}(V).$$

As geodesic flow lines intersect V in segments of length δ_2, the number of times a geodesic flow line intersects V is then

$$\lim_{T\to\infty} \frac{1}{T} (\#\text{segments of } \gamma(0,T) \cap V) = \frac{1}{\delta_2} \mathbf{vol}_{T^1(S_h)}(V).$$

Each segment of $\gamma \cap V$ is within distance at most ϵ_2 of a segment of $\gamma \cap N_{\delta_1}(\beta)$ of length at least $L - \epsilon_2$. Therefore, for any interval $\gamma(I)$ containing the segment, the nearest point projection of $\gamma(I)$ to α has length at least $L - \epsilon_2 - 2\delta_1$. By Proposition 45, the nearest point projection of $\overline{\gamma}(\overline{I})$ to $\overline{\alpha}$ is therefore at least $\epsilon_3 = \frac{1}{Q}(L - \epsilon_2 - 2\delta_1) - c$. As both ϵ_2 and δ_1 are at most δ_3, our choice of L from (2) implies $\epsilon_3 \geqslant 8\delta_3 + 1 > 0$.

By Proposition 24, there is a subset of $\overline{\gamma}$ of diameter $\frac{1}{Q}L - c$ contained in an $6\delta_3$-neighborhood of $\overline{\alpha}$. By Proposition 41, $\overline{\alpha}$ is contained in a K_α-neighborhood of S_0. The nearest point projection map from $\iota(\gamma)$ to $\overline{\gamma}$ is distance decreasing, so $\overline{\gamma}$ spends a positive proportion of its length within distance $K_\alpha + 6\delta_3$ of S_0, as required. □

We show that for almost all geodesics chosen according to Lebesgue measure on $\partial(\widetilde{S}_h \times \mathbb{R})$, the proportion of their length which lies close to S_0 tends to zero. This gives the Lebesgue measure case of Theorem 8.

Lemma 48. *Suppose that $f \colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that ν is the Lebesgue measure on $\partial(\widetilde{S}_h \times \mathbb{R})$. Then for any constant $R > 0$, for $(\nu \times \nu)$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$,*

$$\lim_{T\to\infty} \frac{1}{T} |\, \{ t \in [0,T] \mid \overline{\gamma}(t) \in N_R(S_0) \} \,| = 0.$$

We deduce Lemma 48 directly from the work of Oh and Pan [OP19], which we now describe. We state a special case of their main result which will suffice here. The fibering $M \to S^1$ induces a homomorphism $\pi_1(M) \to \mathbb{Z}$, whose kernel is isomorphic to $\pi_1(S)$. Corresponding to the kernel, we get the $\mathbb{Z}$-cover $M_\mathbb{Z}$ homeomorphic to $S \times \mathbb{R}$. The hyperbolic metric on M lifts to a $\mathbb{Z}$-periodic hyperbolic metric on $M_\mathbb{Z}$.

The hyperbolic metric on M gives a cocompact lattice Γ_0 in $G = \mathrm{PSL}(2, \mathbb{C})$, so that $\Gamma_0 \backslash G$ is the frame bundle for M. The frame flow is denoted by right multiplication by

$a_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$, which projects to the geodesic flow in $\mathbb{H}^3$. Haar measure is a left-invariant measure on G, and determines a measure on $\Gamma_0\backslash G$. The frame flow is ergodic with respect to Haar measure, which projects to the Liouville measure on $T^1(M)$. Oh and Pan show the following mixing result for the geodesic flow.

Theorem 49. [OP19, Theorem 1.7.] *Let Γ_0 be a cocompact lattice in $G = PSL(2, \mathbb{C})$, and let $\Gamma\backslash G$ be a $\mathbb{Z}$-cover of $\Gamma_0\backslash G$. Let ψ_1 and ψ_2 be continuous functions on $\Gamma\backslash G$ with compact support. Then*

$$\lim_{t\to+\infty} t^{1/2} \int_{\Gamma\backslash G} \psi_1(xa_t)\psi_2(x)\ dx = \frac{1}{(2\pi\sigma)^{1/2}} \int_{\Gamma\backslash G} \psi_1\ dx \int_{\Gamma\backslash G} \psi_2\ dx, \tag{3}$$

where σ is a constant depending on Γ_0.

In particular, the above result implies that the proportion of geodesics starting at height zero (in $M_{\mathbb{Z}}$) that are close to height zero at time t decays at rate $1/\sqrt{t}$. This implies that a typical Lebesgue geodesic in $M_{\mathbb{Z}}$ is recurrent on the fibers, but the proportion of time spent near a fixed fiber decays like $1/\sqrt{t}$.

Proof of Lemma 48. Any choice of homeomorphism g from the mapping torus M to the hyperbolic manifold $\mathbb{H}^3/\Gamma$ lifts to a quasi-isometry $\widetilde{g}$ from $\widetilde{S}_h \times \mathbb{R}$ with the Cannon-Thurston metric to $\mathbb{H}^3$ with the standard metric. We shall write $d_{\mathbb{H}^3}$ for the pullback of the hyperbolic metric to $\widetilde{S}_h \times \mathbb{R}$ by $\widetilde{g}$.

Suppose that p is a basepoint at height zero in $\widetilde{S}_h \times \mathbb{R}$. Let ϕ be a rotationally symmetric continuous approximation to the indicator function of p, normalized so that it integrates to one, with respect to the hyperbolic metric $d_{\mathbb{H}^3}$. Let ϕ' be the pull back of ϕ to $G = \mathrm{PSL}(2, \mathbb{C})$. Let ϕ'_t be the composition of ϕ' with the frame flow, that is $\phi'_t(x) = \phi'(xa_t)$. Then the forward and backward projections of ϕ'_t to $\widetilde{S}_h \times \mathbb{R}$ converge to Lebesgue measure on the sphere at infinity $\partial(\widetilde{S}_h \times \mathbb{R})$.

In applying Theorem 49, we set ψ_1 to be the push forward of ϕ' to $\Gamma\backslash G$. We set ψ_2 to be a close approximation of the indicator function for the pre-image of height-zero fiber in $\Gamma\backslash G$. With this choice, the left hand side integral in (3) gives the proportion of geodesics which at time t lie close to the base fiber in $M_{\mathbb{Z}}$. By (3), this proportion goes to zero at rate $1/\sqrt{t}$. As the hyperbolic metric $d_{\mathbb{H}^3}$ is quasi-isometric to the Cannon-Thurston metric $d_{\widetilde{S}_h \times \mathbb{R}}$, this also holds for the Cannon-Thurston metric. $\square$

3.3 Hitting measure

In this section we consider geodesics chosen according to hitting measure determined by random walks.

We start by showing that for the pushforward of a hitting measure on $\partial\widetilde{S}_h$ arising from a random walk, almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$ spend a positive proportion of time close to the base fiber S_0. This completes the proof of Theorem 6 by showing the hitting measure case for surface measures.

Lemma 50. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that ν and $\check{\nu}$ are the forward and backward hitting measures on $\partial \widetilde{S}_h$ arising from a nonelementary, full random walk on $\pi_1(S)$. Let $\iota_* \nu$ and $\iota_* \check{\nu}$ be their pushforwards under the Cannon-Thurston map. Then there are constants $R \geqslant 0$ and $\epsilon > 0$ such that for $(\iota_* \nu \times \iota_* \check{\nu})$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$,*

$$\lim_{T \to \infty} \frac{1}{T} |\{ t \in [0, T] \mid \overline{\gamma}(t) \in N_R(S_0) \}| \geqslant \epsilon.$$

Proof. Let γ be a bi-infinite geodesic arising from the limit points of a bi-infinite random walk. We fix a constant $L > 0$ such that $\frac{L}{Q} - c > 8\delta_3$, where Q and c are constants in Proposition 45. Suppose that g is a non-trivial element of $\pi_1(S)$, with axis α in $\widetilde{S}_h$, and axis $\overline{\alpha}$ in $\widetilde{S}_h \times \mathbb{R}$. As open sets have positive hitting measure, there is a positive probability that the length of the projection interval $p_\alpha(\gamma)$ in $\widetilde{S}_h$ has length at least L. By Proposition 45, the diameter of the projection image $p_{\overline{\alpha}}(\overline{\gamma})$ in $\widetilde{S}_h \times \mathbb{R}$ has diameter at least $\frac{1}{Q_\alpha} L - c$. As $\frac{1}{Q_\alpha} L_1 - c > 8\delta_3$, Proposition 24 implies the projection image $p_{\overline{\alpha}}(\overline{\gamma})$ is contained in a $6\delta_3$-neighborhood of $\overline{\gamma}$. So there are points on $\overline{\gamma}$ distance $\frac{1}{Q_\alpha} L_1 - c - 8\delta_3$ apart, such that the interval between them is contained in a $6\delta_3$-neighborhood of $\overline{\alpha}$. By ergodicity, this happens linearly often for translates of α. As the projection map $p_{\overline{\gamma}}$ from $\iota(\gamma)$ to $\overline{\gamma}$ is distance decreasing, $\overline{\gamma}$ spends a positive proportion of time within distance $6\delta_3$ of S_0. $\square$

We now show that for hitting measure on $\partial(\widetilde{S}_h \times \mathbb{R})$ arising from a geometric random walk on $\pi_1(M)$, for almost all geodesics in $\widetilde{S}_h \times \mathbb{R}$, the proportion of their length which is close to the base fiber S_0 tends to zero. This completes the proof of Theorem 8 by showing the hitting measure case for 3-manifold measures. Theorems 6 and 8 then imply Theorem 4.

Proposition 51. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose ν and $\check{\nu}$ are the forward and backward hitting measures on $\partial(\widetilde{S}_h \times \mathbb{R})$ arising from a geometric random walk on $\pi_1(M)$. Then for $(\nu \times \check{\nu})$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$, and any constant $R > 0$,*

$$\lim_{T \to \infty} \frac{1}{T} |\{ t \in [0, T] \mid \overline{\gamma}(t) \in N_R(S_0) \}| = 0.$$

We shall use the following estimate, which is a consequence of the Local Central Limit Theorem for random walks on $\mathbb{Z}$, see for example [LL10, Section 2].

Proposition 52. [LL10, Proposition 2.4.4] *Suppose $\phi_* \mu$ generates an aperiodic random walk on $\mathbb{Z}$. Then there is a constant C such that for all n and x,*

$$\mathbb{P}(\phi(w_n) = x) \leqslant \frac{C}{\sqrt{n}}.$$

Proof of Proposition 51. We sketch the key steps before giving the technical details.

- By the epimorphism $\phi : \pi_1(M) \to \mathbb{Z}$, the μ-random walk projects to an aperiodic random walk on $\mathbb{Z}$.
- By Proposition 52, the random walk on $\mathbb{Z}$ recurs to $O(\log n)$ neighborhoods of zero with negligible probability (as $n \to \infty$).
- The epimorphism ϕ is distance non-increasing for the Cannon-Thurston metric. So we deduce the same statement for recurrence of μ-random walk to $O(\log n)$ neighborhoods of S_0.
- At the same time, the μ-random walk recurs to a $O(\log n)$ neighborhoods of the tracked geodesic with probability negligibly smaller than 1.
- Choosing the size of the neighborhood of the tracked geodesic a definite proportion smaller than the neighborhood of S_0, we derive the fact that the closest points on the tracked geodesic stay $O(\log n)$ distance away from S_0 with probability negligibly smaller than 1.
- The probability estimates imply a sub-linear upper bound for the expectation of the number of times the tracked geodesic recurs to a $O(\log n)$ neighborhood of S_0.
- Using the expectation bound and linear progress with exponential decay, we conclude the proof of Proposition 51.

Let $\phi_*\mu$ denote the pushforward of μ by the homomorphism $\phi\colon \pi_1(M) \to \mathbb{Z}$. The μ-random walk projects to a $\phi_*\mu$ random walk on $\mathbb{Z}$. As the support of μ generates $\pi_1(M)$ as a semigroup, the support of $\phi_*\mu$ generates $\mathbb{Z}$ as a semigroup. In particular, by Proposition 52, there is a constant C_1 such that for all n and any $A > 0$,

$$\mathbb{P}(|\phi(w_n)| \leqslant 2A \log n) \leqslant 2AC_1 \log n/\sqrt{n}.$$

As the above inequality holds for all $A > 0$, we may choose

$$A = \frac{1}{2 \log \frac{1}{c}}, \tag{4}$$

where $c < 1$ is the exponential decay constant from Proposition 32.

As the distance between any pair of adjacent fibers $S \times \{n\}$ and $S \times \{n+1\}$ is equal to one in the Cannon-Thurston metric, $|\phi(w_n)| \geqslant 2A \log n$ implies that $d_{\widetilde{S}_h \times \mathbb{R}}(w_n x_0, S_0) \geqslant 2A \log n$. In particular, it is very likely that $w_n x_0$ is far from S_0, i.e.

$$\mathbb{P}\left(d_{\widetilde{S}_h \times \mathbb{R}}(w_n x_0, S_0) \geqslant 2A \log n\right) \geqslant 1 - \frac{2AC_1 \log n}{\sqrt{n}}. \tag{5}$$

Recall that $\gamma(t_n)$ is a closest point on γ to $w_n x_0$. The Gromov product estimate, Proposition 32, implies that it is likely that $w_n x_0$ is logarithmically close to γ, i.e.

$$\mathbb{P}\left(d_{\widetilde{S}_h\times\mathbb{R}}(w_n x_0, \gamma(t_n)) \leqslant A\log n\right) \geqslant 1 - Kc^{A\log n}. \tag{6}$$

Combining (5) and (6) above, and using the triangle inequality, shows that it is likely that $\gamma(t_n)$ is far from S_0, i.e.

$$\mathbb{P}\left(d_{\widetilde{S}_h\times\mathbb{R}}(\gamma(t_n), S_0) \geqslant A\log n\right) \geqslant 1 - \frac{2AC_1\log n}{\sqrt{n}} - Kn^{A\log c}.$$

Taking the complementary event in the line above shows that it is unlikely that $\gamma(t_n)$ is close to S_0, and using the fact that our choice of A from (4) makes $A\log\frac{1}{c} = 1/2$, gives

$$\mathbb{P}\left(d_{\widetilde{S}_h\times\mathbb{R}}(\gamma(t_n), S_0) \leqslant A\log n\right) \leqslant \frac{2AC_1\log n + K}{\sqrt{n}}.$$

Let X_n be the number of times $\gamma(t_k)$ is within distance $A\log k$ of S_0 for $1 \leqslant k \leqslant n$. Then an elementary integral comparison bound says that there is a constant C_2 such that

$$\mathbb{E}(X_n) \leqslant 2AC_1\sqrt{n}\log n + 2K\sqrt{n} + C_2.$$

For any random variable, the Markov inequality says $\mathbb{P}(X \geqslant t) \leqslant \mathbb{E}(X)/t$, so choosing $t = \sqrt{n}(\log n)^2$ gives

$$\mathbb{P}(X_n \geqslant \sqrt{n}(\log n)^2) \leqslant \frac{2AC_1}{\log n} + \frac{2K}{(\log n)^2} + \frac{C_2}{\sqrt{n}(\log n)^2}. \tag{7}$$

Let $\beta_n = \gamma([t_{n-1}, t_n])$, and set

$$B_n = \bigcup_{1\leqslant k\leqslant n} \beta_k.$$

Recall that by Corollary 35, t_n makes linear progress with exponential decay. In particular, there is a constant $\ell > 0$ such that

$$\mathbb{P}\left(\gamma([0,\ell n]) \subseteq B_n \subseteq \gamma([0, 2\ell n])\right) \geqslant 1 - Kc^n.$$

By Proposition 36, there is a constant $D > 0$ such that the probability that $|\beta_k| \leqslant D\log k$ for all $k \geqslant \log n$ tends to one as n tends to infinity. Let D_n be the union of all β_k, for $1 \leqslant k \leqslant n$, such that any point on β_k is within distance $(A + D)\log k$ of S_0. By (7), the number of such intervals is at most $\sqrt{n}(\log n)^2$, with probability that tends to one as n tends to infinity. The probability that the union of the first $\log n$ segments $B_{\log n}$ has total length at most $2\ell\log n$ tends to one as n tends to infinity. Therefore,

$$\mathbb{P}\left(\frac{|D_n|}{|B_n|} \leqslant \frac{2\ell\log n + D\sqrt{n}(\log n)^3}{\ell n}\right) \to 1 \text{ as } n \to \infty.$$

In particular, the proportion of points in $\gamma([0, t_n])$ which lie within distance K of S_0 tends to zero as n tends to infinity, as required. $\square$

4 Effective bounds for surface measures

In this section, we prove Theorem 7 using several results that we state and prove in this section, and using our construction of quasigeodesics in $\widetilde{S}_h \times \mathbb{R}$ in our companion paper [GMPU25].

Suppose that γ is a non-exceptional geodesic with unit speed parametrization. The inclusion map ι embeds γ in $\widetilde{S}_h \times \mathbb{R}$ at height $z = 0$. The image $\iota(\gamma)$ is, in general, not a quasigeodesic. However, we will show that the height for each point of $\iota(\gamma)$ can be changed in a specified way so that the resulting path is a quasigeodesic, i.e. there is a function $h_\gamma(t)$ such that $(\gamma(t), h_\gamma(t))$ is an (unparametrized) quasigeodesic.

In order to define the function h_γ, we need the following mild generalization of a measured lamination. A measured lamination is *maximal* if every complementary region is an ideal triangle. By adding finitely many leaves to divide every ideal complementary region of a measured lamination Λ into ideal triangles we may extend Λ to a maximal lamination. There are thus only finitely many such maximal laminations containing Λ. We will call the union of these minimal laminations the *extended lamination* $\overline{\Lambda}$, which in general is not itself a measured lamination. See [GMPU25, Section 2.2] for more details.

Let $\overline{\Lambda}$ be the union of the two extended laminations obtained from the invariant laminations for f, and let $\overline{\Lambda}^1$ be its lift in $T^1(S_h)$. Let $h\colon T^1(S_h) \setminus \overline{\Lambda}^1 \to \mathbb{R}$ be a continuous function. Then h determines an embedding in $\widetilde{S}_h \times \mathbb{R}$ of any non-exceptional unit-speed geodesic γ by the map $t \mapsto (\gamma(t), h(\gamma^1(t)))$, where $\gamma(t)$ is the oriented geodesic and $\gamma^1(t)$ is the unit tangent vector to γ at $\gamma(t)$. We shall call h the *height function*, and the embedding $\tau_\gamma(t) = (\gamma(t), h(\gamma^1(t)))$ the *test path* for γ determined by h.

We now specify the height function we will use, see [GMPU25, Section 3] for background and motivation.

We shall write $\log_k$ for the logarithm function with base k, where $k = k_f > 1$ is the stretch factor of f.

Definition 53. Suppose that $f\colon S \to S$ is a pseudo-Anosov map, let (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations, and $\overline{\Lambda}^1_-$ and $\overline{\Lambda}^1_+$ the lifts in $T^1(S_h)$ of the extended laminations given by the invariant laminations of f, and let $\theta > 0$ be a positive constant. We define the *height function* $h_\theta\colon T^1(S_h) \setminus (\overline{\Lambda}^1_+ \cup \overline{\Lambda}^1_-) \to \mathbb{R}$ to be

$$h_\theta(v) = \log_k \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(v, \overline{\Lambda}^1_+)} - \log \frac{1}{\theta} \right\rfloor_1 - \log_k \left\lfloor \log \frac{1}{d_{\mathrm{PSL}(2,\mathbb{R})}(v, \overline{\Lambda}^1_-)} - \log \frac{1}{\theta} \right\rfloor_1$$

Here $\lfloor x \rfloor_c = \max\{x, c\}$ is the standard floor function. As the two extended laminations are a positive distance apart in $T^1(S_h)$, for sufficiently small θ, at most one of the terms on the right hand side above will be non-zero.

We prove that for a choice of θ sufficiently small, the test path determined by the corresponding height function is a quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$.

Theorem 54. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there are constants $\theta > 0$, $Q \geqslant 1$ and $c \geqslant 0$, such that for any non-exceptional geodesic γ in S_h, with a unit speed parametrization $\gamma(t)$, the test path $\tau_\gamma(t) = (\gamma(t), h_\theta(\gamma^1(t)))$ is an unparametrized (Q, c)-quasigeodesic in $\widetilde{S}_h \times \mathbb{R}$ with the same limit points as $\iota(\gamma)$, where h_θ is the height function from Definition 53.*

We shall now fix a sufficiently small constant θ in Theorem 54 and simplify notation to just write h for h_θ. See [GMPU25, Section 3.2] for the exact choice of θ that we use. Furthermore, we will write $h_\gamma(t)$ for $h_\theta(\gamma^1(t))$.

We will use one further property of these quasigeodesics. The test path τ_γ lies in the ladder $F(\gamma)$ determined by γ, so vertical projection gives a map $(\gamma(t), 0) \mapsto (\gamma(t), h_\gamma(t))$ from $\iota(\gamma)$ to the test path τ_γ. We will prove that this map is coarsely distance non-increasing.

Proposition 55. *Suppose that $f\colon S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. There are constants $K > 0$ and $c \geqslant 0$ such that for any non-exceptional geodesic γ with unit speed parametrization, and any real numbers s and t,*

$$d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(s), \tau_\gamma(t)) \leqslant K d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(s)), \iota(\gamma(t))) + c,$$

where here the test path has the parametrization inherited from the unit speed parametrization on γ.

4.1 Lebesgue measure

Assuming Theorem 54 and Proposition 55, we now derive Theorem 56 giving effective bounds for the amount of time that a geodesic chosen according to Lebesgue measure on the boundary circle of $\widetilde{S}_h$ spends close to the base fiber S_0. This implies the first half of Theorem 7.

Theorem 56. *Suppose that f is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Let ν be the pushforward of Lebesgue measure on $\partial\widetilde{S}_h$ under the Cannon-Thurston map. Then there are constants $K > 0$ and $\alpha > 0$ such that for $(\nu \times \nu)$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$, for any $R \geqslant 0$,*

$$\lim_{T \to \infty} \frac{1}{T} \left| \overline{\gamma}([0, T]) \cap N_R(S_0) \right| \geqslant 1 - K e^{-\alpha k^R}.$$

Here is an overview of the argument.

1. Let $\overline{\Lambda}^1$ be the union of the lifts of both extended laminations in $T^1(S_h)$. By [GMPU25, Definition 26] of the height function h_γ, if $|h_\gamma(t)| \geqslant R$, then the tangent vector at $\gamma(t)$ lies in $N_r(\overline{\Lambda}^1)$, for $r = K e^{-k^R}$.

2. Work of Birman–Series [BS85] shows that the volume of $N_r(\overline{\Lambda}^1)$ goes to zero as $r \to 0$ at the rate $r^2(\log \frac{1}{r})^{6g-6}$.

3. Suppose that γ is a geodesic chosen according to Lebesgue measure on $\partial \widetilde{S}_h$. The geodesic flow on $T^1(S_h)$ is ergodic with respect to Liouville measure, which is the product of the hyperbolic metric on S_h with Lebesgue measure on the unit tangent circles. Therefore almost all geodesics γ are uniformly distributed in $T^1(S_h)$. In particular, we deduce that

   - we may fix R_0 large enough and hence $r_0 = Ke^{-kR_0}$ small enough such that γ recur to $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, and
   - the proportion of time along γ for which $|h_\gamma(t)|$ is at least R is at most $O(e^{-k^R})$, where the time is being measured in terms of a unit speed parametrization of γ in $\widetilde{S}_h$.

4. By the work of Gadre–Hensel [GH24] the distance in $\widetilde{S}_h \times \mathbb{R}$ from $\iota(\gamma)(0)$ to $\iota(\gamma)(T)$ grows linearly in T. From the linear growth and recurrence to $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, we deduce that the distance $d(t)$ between $\tau_\gamma(0)$ and $\tau_\gamma(t)$ also grows linearly in T. From this, we deduce that, as a proportion of $d(T)$, the time along τ_γ for which $|h_\gamma|$ is at least R is again at most $O(e^{-k^R})$.

5. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ. As the test path τ_γ is quasigeodesic, there is a constant L such that the test path is contained in an L-neighborhood of $\overline{\gamma}$. Therefore, parametrizing $\overline{\gamma}$ with unit speed, the proportion of times on $\overline{\gamma}$ outside $N_R(S_0)$ goes to zero at rate $e^{-k^{R-L}}$, as desired.

Step 1 requires no further elaboration. We now justify Step 2.

Let Λ be a geodesic lamination in the surface S_h, and let $N_r(\Lambda)$ be the set of all points in S distance at most r from Λ. Birman and Series [BS85] give the following bounds for the area of $N_r(\Lambda)$.

Theorem 57. [BS85] *Suppose that S is a surface with a complete finite-area hyperbolic metric. Then there are constants $A > 0$ and $r_0 > 0$, such that for any geodesic lamination Λ on S, and for all $r \leqslant r_0$,*

$$\frac{1}{A} r \leqslant \mathbf{area}(N_r(\Lambda)) \leqslant Ar(\log \tfrac{1}{r})^{6g-6}.$$

The upper bound follows from [BS85, Proposition 4.1] with the degree of the exponent given by [BS85, Remark 7.2]. Birman–Series state only the upper bound. The lower bound is immediate from the observation that any geodesic lamination contains an embedded simple arc, and the area of an r-neighborhood of a simple arc is proportional to r for r sufficiently small. The theorem above gives the following immediate bounds on the volumes of an r-neighborhood of Λ in the unit tangent bundle. Let Λ^1 be the pre-image of Λ in the unit tangent bundle $T^1(S)$, and let $N_r(\Lambda^1)$ be the set of all points in $T^1(S)$ distance at most r from Λ^1.

Corollary 58. *Suppose that S is a surface with a complete finite-area hyperbolic metric. Then there are constants $A > 0$ and $r_0 > 0$, such that for any extended geodesic lamination $\overline{\Lambda}$ on S, and all $r \leqslant r_0$,*

$$\frac{1}{A} r^2 \leqslant \mathbf{vol}(N_r(\overline{\Lambda}^1)) \leqslant A r^2 (\log \tfrac{1}{r})^{6g-6}.$$

Proof. A geodesic lamination Λ has finitely many complementary regions and can be extended in finitely many ways to a maximal lamination Λ_i by dividing each complementary region into ideal triangles. The extended lamination $\overline{\Lambda}$ is thus contained in the union of finitely many laminations Λ_i. The result follows immediately from Theorem 57, by replacing A by An, where n is the number of geodesic laminations in the collection Λ_i. □

Step 3 follows from Corollary 58 and ergodicity of the geodesic flow on $T^1(S_h)$. Using [GMPU25, Definition 26], we may write a version of Theorem 56 for the test path τ_γ, using the parametrization of γ in $\widetilde{S}_h$, instead of a unit speed parametrization of the test path.

Proposition 59. *Suppose f is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there is a constant $K > 0$ such that for Lebesgue-almost all geodesics γ in $\widetilde{S}_h$, for any unit speed parametrization of the geodesic $\gamma(t)$, for any $R \geqslant 0$,*

$$\lim_{T\to\infty} \frac{1}{T} \left| \tau_\gamma([0,T]) \setminus N_R(S_0) \right| \leqslant K e^{-k^R},$$

where $\tau_\gamma(t)$ is the parametrization induced by $\gamma(t)$, not the arc length parametrization.

Proof. By ergodicity of the geodesic flow, the amount of time almost every geodesic spends within distance r of the union of the extended laminations $\overline{\Lambda}^1$ is equal to the volume of $N_r(\overline{\Lambda}^1)$.

$$\lim_{T\to\infty} \frac{1}{T} \left| \gamma^1([0,T]) \cap N_r(\overline{\Lambda}^1) \right| = \mathbf{vol}(N_r(\overline{\Lambda}^1))$$

By Corollary 58,

$$\lim_{T\to\infty} \frac{1}{T} \left| \gamma^1([0,T]) \cap N_r(\overline{\Lambda}^1) \right| \leqslant A r^2 (\log \tfrac{1}{r})^{6g-6}. \tag{8}$$

By the definition of the test path, [GMPU25, Theorem 27], if $\gamma(t)$ lies in $N_r(\overline{\Lambda}^1)$ then the corresponding point on the test path $\tau_\gamma(t)$ lies outside $N_R(S_0)$, where $r = K e^{-k^R}$. Rewriting the upper bound in (8) in terms of R gives

$$\lim_{T\to\infty} \frac{1}{T} \left| \tau_\gamma([0,T]) \setminus N_R(S_0) \right| \leqslant A K^2 e^{-2k^R} k^{(6g-6)R}.$$

Furthermore, for sufficiently large R, $e^{k^R} \geqslant k^{(6g-6)R}$, so for an appropriate choice of $K_1 = AK^2$,

$$\lim_{T\to\infty} \frac{1}{T} |\tau_\gamma([0,T]) \setminus N_R(S_0)| \leqslant K_1 e^{-k^R}, \tag{9}$$

as required. □

To deduce Step 4, we first state the following result of Gadre–Hensel [GH24] showing that the distance in $\widetilde{S}_h \times \mathbb{R}$ along $\iota(\gamma)$ grows linearly.

Theorem 60. [GH24, Theorem 2.2] *Suppose that $f: S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there is a constant $\epsilon > 0$ such that for Lebesgue-almost all geodesics γ in $\widetilde{S}_h$, with unit speed parametrization, we have*

$$\frac{1}{T} d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(0)), \iota(\gamma(T))) > \epsilon$$

for all T large enough depending on γ.

We deduce from Theorem 60, the recurrence of γ to $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, and the quasi-geodesicity of τ_γ, that the distance in $\widetilde{S}_h \times \mathbb{R}$ along the test path τ_γ also grows linearly with respect to the parametrization coming from γ.

Corollary 61. *Suppose that $f: S \to S$ is a pseudo-Anosov map and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Then there is a constant $\epsilon > 0$ such that for Lebesgue-almost all geodesics γ in $\widetilde{S}_h$, with unit speed parametrization, we have*

$$\frac{1}{T} d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(0), \tau_\gamma(T)) > \epsilon.$$

for all T large enough depending on γ.

Proof. Suppose that γ is a non-exceptional geodesic in $\widetilde{S}_h$ sampled with respect to the Lebesgue measure and parametrized by unit speed. Suppose that the test path τ_γ is given the parametrization from γ.

We may fix R_0 large enough and hence $r_0 = Ke^{-kR_0}$ small enough such that for Lebesgue-almost all γ the time that $\gamma([0,T])$ spends in $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$ is strictly greater than $2/3$ for all T large enough. Suppose that $t_0 > 0$ is the smallest time for which $\gamma(t_0)$ lies in $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$, and suppose that $t_1 < T$ is the largest time for which $\gamma(t_1)$ lies in $T^1(S_h) \setminus N_{r_0}(\overline{\Lambda}^1)$. It follows that $t_0 < T/3$ and $t_1 > 2T/3$, and so $t_1 - t_0 \geqslant T/3$. By the triangle inequality,

$$d_{\widetilde{S}_h \times \mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_1)) \geqslant d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \iota(\gamma(t_1))) - d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_0)), \tau_\gamma(t_0)) - d_{\widetilde{S}_h \times \mathbb{R}}(\iota(\gamma(t_1)), \tau_\gamma(t_1)).$$

By definition of the test path, the final two terms on the right hand side are equal to the absolute value of the height function, and so

$$\begin{aligned} d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0),\tau_\gamma(t_1)) &\geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\iota(\gamma(t_0)),\iota(\gamma(t_1))) - |h_\gamma(t_0)| - |h_\gamma(t_1)| \\ &\geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\iota(\gamma(t_0)),\iota(\gamma(t_1))) - R_0 - R_0. \end{aligned}$$

By Theorem 60, $d_{\widetilde{S}_h\times\mathbb{R}}(\iota(\gamma(t_0)),\iota(\gamma(t_1))) \geqslant \epsilon(t_1-t_0)$, and so

$$d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0),\tau_\gamma(t_1)) \geqslant \epsilon(t_1-t_0) - 2R_0 \geqslant \frac{1}{3}\epsilon T - 2R_0.$$

We conclude the proof by noting that the test path τ_γ is a quasi-geodesic in $\widetilde{S}_h\times\mathbb{R}$ and hence the distance $d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0),\tau_\gamma(t_1))$ is a coarse lower bound for $d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(0),\tau_\gamma(T))$. $\square$

Finally, we prove Step 5, completing the proof of Theorem 56.

Proof (of Theorem 56). We obtain from γ a parametrization $\mathbb{R}\to\overline{\gamma}$ by letting $\overline{\gamma}_t$ be a point of $\overline{\gamma}$ closest to $\tau_\gamma(t)$. Thus $\overline{\gamma}_0$ and $\overline{\gamma}_T$ are points of $\overline{\gamma}$ closest to $\tau_\gamma(0)$ and $\tau_\gamma(T)$ respectively. Since τ_γ is a quasi-geodesic there is a constant L such that $d_{\widetilde{S}_h\times\mathbb{R}}(\overline{\gamma}_0,\tau_\gamma(0)) < L$ and $d_{\widetilde{S}_h\times\mathbb{R}}(\overline{\gamma}_T,\tau_\gamma(T)) < L$. By triangle inequality,

$$d_{\widetilde{S}_h\times\mathbb{R}}(\overline{\gamma}_0,\overline{\gamma}_T) \geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(0),\tau_\gamma(T)) - 2L.$$

By Corollary 61, there is then an $\epsilon>0$ such that

$$d_{\widetilde{S}_h\times\mathbb{R}}(\overline{\gamma}_0,\overline{\gamma}_T) \geqslant \epsilon T \tag{10}$$

On the other hand, the projection $\iota(\gamma)$ to τ_γ along flow lines is distance decreasing. Hence, it follows that

$$d_{\widetilde{S}_h\times\mathbb{R}}(\overline{\gamma}_0,\overline{\gamma}_T) \leqslant T + 2L. \tag{11}$$

By Proposition 59, for Lebesgue almost all geodesics γ, there is a $T_1(\gamma)$, such that for all $T\geqslant T_1(\gamma)$,

$$|\tau_\gamma([0,T]) \setminus N_R(S_0)| \leqslant TKe^{-k^R}.$$

As $\overline{\gamma}$ is contained in an L-neighborhood of the test path τ_γ, we deduce

$$|\overline{\gamma}([0,T]) \setminus N_{R+L}(S_0)| \leqslant TKe^{-k^R}$$

where $\overline{\gamma}$ is parametrized from γ. Finally, suppose that $\overline{\gamma}(D) = \overline{\gamma}_T$ in the arc length parametrization of $\overline{\gamma}$ in which $\overline{\gamma}(0) = \overline{\gamma}_0$. Then, by Equation (10),

$$|\overline{\gamma}([0,T]) \setminus N_{R+L}(S_0)| \leqslant \frac{1}{\epsilon}DKe^{-k^R}$$

from which, by tweaking constants, we may deduce Theorem 56, as required. $\square$

4.2 Hitting measure

Again assuming Theorem 54 and Proposition 55, we now derive Theorem 62 giving effective bounds for the proportion of time that a geodesic chosen by a hitting measure on the boundary circle of $\widetilde{S}_h$ spends close to the base fiber S_0. This completes the second half of Theorem 7.

Theorem 62. *Suppose that f is a pseudo-Anosov map with stretch factor $k > 1$, and (S_h, Λ) is a hyperbolic metric on S together with a pair of invariant measured laminations. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let ν and $\check{\nu}$ be the forward and backwards hitting measures on $\partial\widetilde{S}_h$. Then there are constants $K > 0$ and $\alpha > 0$ such that for $(\iota_*\nu \times \iota_*\check{\nu})$-almost all geodesics $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$, for any unit speed parametrization $\overline{\gamma}(t)$, for any $R \geqslant 0$,*

$$\lim_{T\to\infty} \frac{1}{T} |\overline{\gamma}([0,T]) \setminus N_R(S_0)\}| \leqslant K e^{-\alpha k^R}.$$

Here is an overview of the argument.

1. Suppose that γ is a geodesic in $\widetilde{S}_h$ chosen according to the hitting measure $\nu \times \check{\nu}$ on $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Suppose that the tangent vector at $\gamma(t)$ is within distance $r > 0$ of one of the extended laminations Λ. If r is very small then the endpoints of γ are within distance $r + o(r)$ of the limit set of Λ, when $\partial\widetilde{S}_h$ is equipped with the angular metric viewed from $\gamma(t)$. Work of Birman–Series [BS85] shows that the hitting measure of $N_r(\partial\Lambda)$ goes to zero at rate r^α.

2. Let x_0 be a choice of basepoint in $\widetilde{S}_h$, and let $\gamma(t_n)$ be the closest point on γ to $w_n x_0$. The previous argument shows that the probability that the tangent vector at $\gamma(t_n)$ is within distance r of one of the extended laminations goes to zero at rate r^α. Let R be such that $r = e^{-k^R}$. By the definition of the height function, the probability that $|h_\gamma(t_n)| \geqslant R$ goes to zero at rate $e^{-\alpha k^R}$.

3. Since μ is assumed to have finite support, the distance in $\widetilde{S}_h$ between any two successive locations $w_n x_0$ and w_{n+1} of the random walk is bounded. As nearest point projection to geodesics is distance reducing, the distance between any two successive nearest point projections $\gamma(t_n)$ and $\gamma(t_{n+1})$ is also bounded. We use the fact that the height function is Lipschitz to show that that the distance in $\widetilde{S} \times \mathbb{R}$ between the corresponding test path locations $\tau_\gamma(t_n)$ and $\tau_\gamma(t_n + 1)$ is also bounded.

4. We use the linear progress/ drift of the random walk in $\widetilde{S}_h \times \mathbb{R}$ to show that the test path locations $\tau_\gamma(t_n)$ also make linear progress in $\widetilde{S}_h \times \mathbb{R}$. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ. We parametrize $\overline{\gamma}$ by unit speed so that $\overline{\gamma}(0)$ is the point closest to the first test path location $\tau_\gamma(t_0)$. As the test path τ_γ is contained in a bounded neighborhood of the geodesic $\overline{\gamma}$, the number of test path locations close to the segment $\overline{\gamma}([0,T])$ grows linearly in T.

5. As the proportion of test path locations outside of $N_R(S_0)$ goes to zero at rate $e^{-\alpha k^R}$, the proportion of time $\overline{\gamma}([0,T])$ spends outside $N_R(S_0)$ goes to zero at the same rate.

We start by estimating hitting measure for a regular neighborhood of the limit set of the extended laminations. We will use the following result from the proof of [BS85, Theorem II, page 224], with the degree of the polynomial from [BS85, Remark 7.2].

Theorem 63. [BS85] *Suppose that S_h is a closed hyperbolic surface and Λ is a geodesic lamination. Let x_0 be a basepoint in $\widetilde{S}_h$. Then there are constants A, c and $\alpha > 0$ such that for any integer $n > 0$, there are An^{6g-g} squares of side length $ce^{-\alpha n}$ which cover $\partial\Lambda$ in $(\partial S_h \times \partial S_h) \setminus \Delta$.*

We use Theorem 63 to estimate the hitting measure of a regular neighborhood of the limit set of the extended laminations.

Corollary 64. *Suppose that f is a pseudo-Anosov map and S_h a hyperbolic metric. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let $\nu \times \check{\nu}$ be the resulting stationary measure on $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Let $\partial\overline{\Lambda}$ be the limit set in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$ of the union of the extended invariant laminations. Choose a basepoint x_0 in $\widetilde{S}_h$, and give $\widetilde{S}_h$ the visual metric based at x_0, and give $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$ the product metric. Then there are constants $K > 0$ and $\alpha > 0$ such that*

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant Kr^\alpha$$

Proof. As noted before, the union of the extended laminations $\overline{\Lambda}$ is contained in the union of a finite number of geodesic laminations $\Lambda_1, \ldots \Lambda_k$. Let A_1, c_1 and α_1 be the constants from Theorem 63. Setting $r = c_1 e^{-\alpha_1 n}$ in Theorem 63, each $N_r(\Lambda_i)$ is contained in the union of at most $A_1(\frac{1}{\alpha_1}\log\frac{c_1}{r})^{6g-6}$ squares of side length r. As an r-neighborhood of a square of side length r is contained in a union of 9 squares of side length r, this shows that $N_r(\overline{\Lambda})$ is contained in the union of at most $9A_1k(\frac{1}{\alpha}\log\frac{c_1}{r})^{6g-6}$ squares of side length r.

The Gromov product of two points in the boundary is equal to $\log(1/\sin(\theta/2))$, where θ is the angle between them viewed from the basepoint, see for example [Roe03, page 114]. Using the elementary bounds $x/2 \leqslant \sin x \leqslant x$ for $0 \leqslant x \leqslant 1$, we know that an interval of length r in $\partial\widetilde{S}_h$ is contained in a shadow of distance $\log\frac{1}{r}$ from the basepoint. By exponential decay of shadows, Proposition 32, there is a constant $c_2 < 1$ such that the hitting measure of an interval of length r is at most $Kc_2^{\log\frac{1}{r}} = Kr^\beta$ for some $\beta = \log\frac{1}{c_2} > 0$. So the hitting measure of a square of side length r is at most $K^2r^{2\beta}$. This gives

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant K^2r^{2\beta}\ 9A_1(\tfrac{1}{\alpha}\log\tfrac{c_1}{r})^{6g-6}.$$

As $(\log\frac{c_1}{r})^{6g-6}$ is a polynomial in $\log\frac{1}{r}$, there is a constant A_2 such that

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant A_2r^{2\beta}(\log\tfrac{1}{r})^{6g-6}.$$

As $r^\beta(\log \frac{1}{r})^{6g-6}$ tends to zero as r tends to zero, there is a constant A_3 such that

$$\nu \times \check{\nu}(N_r(\partial\overline{\Lambda})) \leqslant A_3 r^\beta,$$

and so the result follows by setting $K = A_3$ and $\alpha = \beta$. $\square$

Let x_0 be a basepoint for $\widetilde{S}_h$. As the distribution μ generating the random walk has finite support, the distance between any two successive locations $w_n x_0$ and $w_{n+1}x_0$ of the random walk is bounded. As closest point projection to a geodesic is distance reducing, this implies that the distance between the corresponding closest points $\gamma(t_n)$ and $\gamma(t_{n+1})$ on γ is also bounded. We now show that the distance between the corresponding test path locations $\tau_\gamma(t_n)$ and $\tau_\gamma(t_{n+1})$ is bounded.

Proposition 65. *Suppose that f is a pseudo-Anosov map and S_h a hyperbolic metric. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let x_0 be a basepoint for $\widetilde{S}_h$. Let γ be the geodesic with the same limit points as $(w_n x_0)_{n\in\mathbb{Z}}$, and let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to γ. Then there is a constant B such that for all n,*

$$d_{\widetilde{S}\times\mathbb{R}}(\tau_\gamma(t_n), \tau_\gamma(t_{n+1})) \leqslant B.$$

Proof. Let x_0 be a basepoint for $\widetilde{S}_h$, and let $(w_n x_0)_{n\in\mathbb{Z}}$ be a bi-infinite sample path of the random walk. Let γ be the geodesic determined by its limit points of the sample path. By a unit speed parametrization of γ, we get a parametrization of the test path $\tau_\gamma(t)$.

As μ has finite support, there is an upper bound B_1 on the distance between $w_n x_0$ and $w_{n+1}x_0$. Let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to the geodesic γ in $\widetilde{S}_h$. As nearest point projection to a geodesic is distance decreasing, the distance between $\gamma(t_n)$ and $\gamma(t_{n+1})$ is at most B_1. As the inclusion map ι is distance decreasing, the distance between $\iota(\gamma(t_n))$ and $\iota(\gamma(t_{n+1}))$ is also at most B_1.

By Proposition 55, there are constants K and c such that the distance between the corresponding test path locations $\tau_\gamma(t_1)$ and $\tau_\gamma(t_2)$ is at most $B = KB_1 + c$, as required. $\square$

We now show that the distance in $\widetilde{S}_h \times \mathbb{R}$ between the test path locations $\tau_\gamma(t_0)$ and $\tau_\gamma(t_n)$ grows linearly in n.

Proposition 66. *Suppose that f is a pseudo-Anosov map and S_h a hyperbolic metric. Suppose that μ is a finitely supported probability measure on $\pi_1(S)$ whose support generates $\pi_1(S)$ as a semigroup. Let x_0 be a basepoint for $\widetilde{S}_h$. Let γ be the geodesic with the same limit points as $\{w_n x_0\}$, and let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to γ. Then there is a constant $\ell > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{N} d_{\widetilde{S}\times\mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_N)) = \ell > 0.$$

Proof. Let $\overline{x}_0 = \iota(x_0)$. By the triangle inequality applied to the path consisting of the three geodesic segments $[\overline{x}_0, \tau_\gamma(t_0)] \cup [\tau_\gamma(t_0), \tau_\gamma(t_n)] \cup [\tau_\gamma(t_n), w_n\overline{x}_n]$, we get

$$d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_n)) \geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, w_n\overline{x}_0)) - d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, \tau_\gamma(t_0)) - d_{\widetilde{S}_h\times\mathbb{R}}(w_n\overline{x}_0, \tau_\gamma(t_n)).$$

Set $A = d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, \tau_\gamma(t_0))$, which is independent of n. As the inclusion map ι is distance decreasing, the distance in $\widetilde{S}_h \times \mathbb{R}$ from $w_n\overline{x}_0$ to $\tau_\gamma(t_n)$ is at most the distance in $\widetilde{S}_h$ from w_nx_0 to $\gamma(t_n)$, plus the distance from $\iota(\gamma(t_n))$ to $\tau_\gamma(t_n)$, which is given by the value of the height function $h_\gamma(t_n)$. This gives

$$d_{\widetilde{S}_h\times\mathbb{R}}(\tau_\gamma(t_0), \tau_\gamma(t_n)) \geqslant d_{\widetilde{S}_h\times\mathbb{R}}(\overline{x}_0, w_n\overline{x}_0)) - A - d_{\widetilde{S}_h}(w_nx_0, \gamma(t_n)) - h_\gamma(t_n).$$

The random walk makes linear progress in $\widetilde{S}_h \times \mathbb{R}$ at rate ℓ_3, with exponential decay. The random variable $d_{\widetilde{S}_h}(x_0, \gamma(t_0))$ has a distribution with exponential tails. By Corollary 64, the random variable $h_\gamma(t_0)$ also has exponential tails. Hence, as $N \to \infty$, $\frac{d_{\widetilde{S}_h}(w_Nx_0,\gamma(t_N))}{N} \to 0$ and $\frac{h_\gamma(t_N)}{N} \to 0$. Therefore

$$\lim_{t\to\infty} \frac{1}{N} d_{\widetilde{S}_h\times\mathbb{R}}\left(\overline{\gamma}(\overline{t}_0), \overline{\gamma}(\overline{t}_N)\right) \geqslant \ell_3 > 0,$$

as required. ∎

We now record the following elementary properties of visual measure in $\widetilde{S}_h$.

Proposition 67. *Suppose that x_0 is a basepoint for $\mathbb{H}^2$, and ℓ a geodesic. Suppose that p is the closest point of ℓ to x_0. For $r \leqslant \frac{1}{2}$, let γ be a geodesic which intersects an r-neighborhood in $T^1(\mathbb{H}^2)$ of the tangent vector to ℓ at p. Then the endpoints of ℓ and γ in $\partial_\infty\mathbb{H}^2$ are distance at most $8r$ apart.*

Proof. Suppose ℓ passes through x_0 so that $p = x_0$. Suppose that γ also passes through x_0 and makes an angle at most r with ℓ. Then the distance between their endpoints in the visual metric on $\partial_\infty\mathbb{H}^2$ from x_0, is at most r.

Now suppose that

- ℓ and γ are disjoint, and
- x_0 is the midpoint of the shortest geodesic segment α between ℓ and γ.

Let β be the perpendicular bisector to α at x_0. Choose an endpoint in $\partial_\infty\mathbb{H}^2$ of γ and let q be the nearest point projection of that endpoint to β. In fact, the geodesic between the endpoint and q extends to a bi-infinite geodesic perpendicular to β at q. Consider the right angled triangle with vertices x_0, q and the chosen endpoint of γ as an ideal vertex. Let t be the distance from x_0 to q. By Proposition 25, $t \geqslant \log \frac{2}{r}$. The angle at x_0 is half the angle between the endpoints of ℓ and γ at x_0. By the tangent formula for right angled hyperbolic triangles, $\tan(\theta/2) = 1/\sinh(t)$. Using the elementary bounds that $x \leqslant \tan(x) \leqslant 2x$ for $0 \leqslant x \leqslant 1$, and the bound on t in terms of r, we get $x \leqslant 8r/(4 - r^2)$. As we have assumed that $r \leqslant \frac{1}{2}$, this implies that $\theta/2 \leqslant 4r$, as required.

Finally, suppose that in the case that the geodesics intersect, x_0 is not the point of intersection, and if the geodesics do not intersect, x_0 is not the midpoint of the shortest geodesic segment between them. In the former case, we may fix some isometry g that moves x_0 to the intersection point. Restricted to small intervals about the endpoints of $g^{-1}\gamma$ and their images by g, the isometry g is a contraction for the visual metric from x_0. In the case that γ and ℓ do not intersect, we consider an isometry that moves x_0 to the midpoint of the shortest geodesic segment between γ and ℓ. Again, the isometry restricts to a contraction near the endpoints of $g^{-1}(\gamma)$, and so the result follows. □

We now prove Theorem 62.

Proof (of Theorem 62). We fix a basepoint x_0 in $\widetilde{S}_h$ and give $\partial\widetilde{S}_h$ the angular metric from x_0. Suppose that $(w_n x_0)_{n\in\mathbb{Z}}$ is a typical bi-infinite sample path of the random walk on $\pi_1(S)$ generated by μ. Let γ be the bi-infinite geodesic determined by the limit points of the sample path. Let $\gamma(t_n)$ be the nearest point projection of $w_n x_0$ to γ in $\widetilde{S}_h$. The value of the height function $h_\gamma(t_n)$ is determined by the distance from the tangent vector at $\gamma(t_n)$ to the invariant laminations in $T^1(S_h)$.

Let θ_Λ be the constant from the definition of the height function, [GMPU25, Section 3.2.2] By Proposition 67, if $\gamma(t)$ is distance at most $r \leqslant \theta_\Lambda \leqslant 1$ in $T^1(S_h)$ from a leaf ℓ of one of the invariant laminations, then the endpoints of γ are distance at most $8r$ from the endpoints of ℓ in ∂S_h. We shall write $\overline{\Lambda}$ for the union of the extended invariant laminations, and $\partial\overline{\Lambda}$ for the boundary in $\partial\widetilde{S}_h \times \partial\widetilde{S}_h$. Then

$$\mathbb{P}\left(d_{T^1(S_h)}(\gamma^1(t_0), \overline{\Lambda}^1) \leqslant r\right) \leqslant \nu \times \check{\nu}\left(N_{8r}(\partial\overline{\Lambda})\right),$$

and by Corollary 64 there are constants A_1 and $\beta > 0$ such that

$$\mathbb{P}\left(d_{T^1(S_h)}(\gamma^1(t_0), \overline{\Lambda}^1) \leqslant r\right) \leqslant A_1 r^\beta.$$

Action by the shift map then gives us the same result for all n.

Using the relation between distance r in $T^1(S_h)$ and the value of the height function R, there is a constant A_2 such that

$$\mathbb{P}(|h_\gamma(t_n)| \geqslant R) \leqslant A_2 e^{-\beta k^R}.$$

We shall choose the basepoint in $\widetilde{S}_h \times \mathbb{R}$ to be $\overline{x}_0 = \iota(x_0)$. Let $\overline{\gamma}$ be the geodesic in $\widetilde{S}_h \times \mathbb{R}$ determined by γ. Let $\tau_\gamma(t_n)$ be the corresponding point on the test path, and let $\overline{\gamma}(p_n)$ be the nearest point projection of $\tau_\gamma(t_n)$ to $\overline{\gamma}$ in $\widetilde{S}_h \times \mathbb{R}$.

By Proposition 66, the distance between $\tau_\gamma(t_0)$ and $\tau_\gamma(t_n)$ grows linearly in n. As τ_γ is contained in an L-neighborhood of $\overline{\gamma}$, the distance between $\overline{\gamma}(p_0)$ and $\overline{\gamma}(p_n)$ also grows linearly in n.

Every point of $\overline{\gamma}([p_0, p_N]) \setminus N_R(S_0)$ is within distance $B_2 = B + 2L$ of a point $\overline{\gamma}(p_k)$, and the proportion of such points is at most $\mathbb{P}(|h_\gamma(t_0)| \geqslant R)$. Therefore

$$\lim_{N\to\infty} \frac{1}{|[p_0, p_N]|} |\overline{\gamma}([p_0, p_N]) \setminus N_R(S_0)| \leqslant \frac{B_2}{\ell_3} A_2 e^{-\beta k^R}.$$

As p_N tends to infinity as N tends to infinity, the result follows. □

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